MTG 6256, Fall 2004: Non-book Problem 1
(corrected 8/27/04)

Definition. For $p \in \mathbf{R}^3$, let $\mathcal{F}_p = \{f \mid f$ is a smooth real-valued function defined on some open neighborhood of $p\}$. Define an equivalence relation $\sim$ on $\mathcal{F}_p$ by declaring $f \sim g$ if there exists an open neighborhood of p on which f and g coincide. An equivalence class under this relation is called a *smooth germ*, and the set $\mathcal{G}_p := \mathcal{F}_p / \sim$ of equivalence classes is called the *space of smooth germs of functions at p*. Note that all elements of a given equivalence class have the same value at p, so if $\hat{f} \in \mathcal{G}_p$, the number $\hat{f}(p)$ is well-defined.

(a) Give an explicit necessary and sufficient condition for a function $f \in \mathcal{F}_p$ to represent the *0-germ*, the equivalence class of the constant function 0. Do the same for the equivalence class of a general constant function (a *constant germ*).

(b) Show that the usual operations of addition of functions and multiplication by scalars induce well-defined operations on germs, and therefore that $\mathcal{G}_p$ is a vector space, whose zero element is the 0-germ, under these operations.

(c) Show that multiplication of functions induces a well-defined operation on germs. (We still call this operation multiplication, and denote it the same way we do mulitiplication of functions.)

Definition. A linear functional $L : \mathcal{G}_p \to \mathbf{R}$ is called *Leibnizian* (or *Leibniz-linear*) if for all $\hat{f}, \hat{g} \in \mathcal{G}_p$ we have $L(\hat{f}\hat{g}) = L(\hat{f})\hat{g}(p) + \hat{f}(p)L(\hat{g})$. ("Functional" is another word for real-valued function that is often used when the domain is some infinite-dimensional object.)

(d) Show that if $L : \mathcal{G}_p \to \mathbf{R}$ is Leibnizian then $L(\text{any constant germ}) = 0$.

Let $\mathcal{L}_p = \{\text{Leibnizian functionals } \mathcal{G}_p \to \mathbf{R}\}$.

(e) Show that $\mathcal{L}_p$ is a vector space (with the obvious operations and zero element).

(f) Show that $\mathcal{L}_p$ is naturally isomorphic to $T_p(\mathbf{R}^3)$. (Hint: Taylor's Theorem [*not* Taylor series]. You have to know the right version of Taylor's Theorem for this hint to be useful; the one most commonly taught in modern Calculus 1-2-3 courses, while pretty, is useless for many purposes, including this one.)