MTG 6256, Fall 2004: Non-book Problem 2
corrected 9/18/04

Recall that an open half-plane in $\mathbf{R}^2$ is a set of the form $\{w \in \mathbf{R}^2 \mid (w-p)\cdot v > 0\}$ where $p, v \in \mathbf{R}^2$ and $v \neq 0$. Any line ℓ in $\mathbf{R}^2$ divides $\mathbf{R}^2 - \ell$, the complement of the line, into two open half-planes.

(a) Let ℓ be a line in $\mathbf{R}^2$. (i) Show that, for any point $p \in \ell$ and any $v \in T_p\mathbf{R}^2$ not tangent to ℓ, the half-planes into which ℓ divides $\mathbf{R}^2 - \ell$ are $\{w \in \mathbf{R}^2 \mid (w-p)\cdot \pi(v) > 0\}$ and $\{w \in \mathbf{R}^2 \mid (w-p)\cdot \pi(v) < 0\}$, where $w-p$ is viewed as an element of $T_p\mathbf{R}^2$ and where π is the orthogonal projection from $T_p\mathbf{R}^2$ onto the subspace normal to ℓ at p. (A vector u is *normal to ℓ at p* if it is orthogonal to some, hence every, nonzero vector tangent to ℓ at p.) (ii) Show that the half-line parametrized by $t \mapsto \gamma(t) := p + tv,\ t > 0$, lies in the first of these two half-planes. Thus we define the *half-plane into which v points* to be this half-plane.

(b) Let $\beta : I \to \mathbf{R}^2$ be a smooth unit-speed curve, let $s_0 \in I$, and let ℓ be the line tangent to the image of β at $\beta(s_0)$. Show that if $\kappa(s_0) > 0$, then for ϵ sufficiently small, if $0 < |s - s_0| < \epsilon$ then $\beta(s)$ lies in the half-plane into which the principal unit normal $N(s_0)$ points.

The real point of this exercise is that it gives a parametrization-independent characterization of which of the two unit vectors normal to a regular plane curve at a given point p is the principal unit normal: it's the one pointing into the half-plane in which the curve lies near p.