MTG 6256, Fall 2004: Non-book Problem 4

Throughout this problem, O is an open subset of $\mathbf{R}^3$.

(a) Let $\{\theta^i\}_{i=1}^3$ be a (generalized) co-frame field on an open subset O, i.e. a triple of 1-forms on O that at each $p \in O$ form a basis of the cotangent space $T_p^*\mathbf{R}^3$. As in class, write

$$\theta := \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix}, \quad dx := \begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \end{pmatrix}$$

Let $\{E_i\}_{i=1}^3$ be the (not necessarily orthonormal) frame-field dual to $\{\theta^i\}_{i=1^3}$ (i.e. for which $\langle \theta^i, E_j \rangle = \delta^i{}_j$), and define $E = (E_1, E_2, E_3)$. Similarly define $U = (U_1, U_2, U_3)$, where $\{U_i\}$ is the standard frame-field on $\mathbf{R}^3$ (restricted to O). Show that if B is the matrix-valued function for which $\theta = B dx$, then $E = UB^{-1}$.

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Let $\{y^i\}_{i=1}^3$ be a coordinate-system on O. By this we mean a set of three functions $y^i : O \to \mathbf{R}$ for which the Jacobian $(\partial y^i/\partial x^j)$ is invertible at every point of O (here $\{x^i\}$ are the standard coordinates on $\mathbf{R}^3$, restricted to O), and for which the map $O \to \mathbf{R}^3$ given by $p \mapsto (y^1(p), y^2(p), y^3(p))$ is one-to-one (so that each point of O has a unique triple of y-coordinates).

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(b) Show at each point $p \in O$, the $\{dy^i|_p\}$ are linearly independent, and hence that the $\{dy^i\}$ form a co-frame field on O.

(c) In class we discussed why, given a coordinate-system $\{y^i\}$, it is reasonable to label the elements of the (generally non-orthonormal) frame-field dual to the co-frame field $\{dy^i\}$ as $\{\partial/\partial y^i\}$. Apply part (a) to give an expression for dy and for $(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3})$ in terms of the Jacobian $(\partial y^i/\partial x^j)$ and the standard objects $dx, (\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$. Your answer (if correct) provides further justification for the partial-derivative notation for coordinate-system frame fields.

(d) The objects $\{dy^i\}$ and $\{\partial/\partial y^i\}$ discussed above are not quite on the same footing. For *any* function f on O, the 1-form df is defined. Thus if $\{y^i\}$ are coordinates on O, each dy^i is an object whose definition, and whose action on functions, is completely independent of the other dy^j; changing y^2 and y^3 into different functions will not affect y^1, and we do not even need y^2 and y^3 to be defined in order to define dy^1. By contrast, $\{\partial/\partial y^1\}$ (or any of the three $\partial/\partial y^i$) becomes well-defined *only* once the entire triple of coordinate-functions $\{y^i\}_{i=1}$ has been defined; given a single function f, there is no unambiguous meaning to "$\partial/\partial f$". The same is true if the dimension "3" is replaced by any $n \geq 2$. Check this with the following two-dimensional example. Let (x, y) be the usual coordinates on $\mathbf{R}^2$ and define $u = x, v = x + y$. Define $g(x, y) = x + y$. [CONTINUED ON NEXT PAGE.]

1. Show that $\{u, v\}$ is a coordinate-system on $\mathbf{R}^2$.

2. Show that $\frac{\partial}{\partial x}[g] = 1$ (identically), but $\frac{\partial}{\partial u}[g] = 0$ (identically), even though $u : \mathbf{R}^2 \to \mathbf{R}$ and $x : \mathbf{R}^2 \to \mathbf{R}$ are the same function. (Here $\frac{\partial}{\partial x}$ means the first vector field in the coordinate frame-field $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$, $\frac{\partial}{\partial u}$ means the first vector field in the coordinate frame-field $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$, and $V[g]$ denotes the usual action of a vector field V on a function f.)