MTG 6256, Fall 2004: Non-book Problem 6

Below, M is a surface in $\mathbf{R}^3$.

(a) Let $F : \mathbf{R}^3 \to \mathbf{R}^3$ be a diffeomorphism and let $N = F(M)$. Prove that N is a surface.

(b) Let F, N be as in (a) and let Z be a vector field along M (not necessarily tangent or normal). Show that the formula

$$(F_*Z)_p := F_{*F^{-1}(p)}(Z_{F^{-1}(p)}), \quad p \in N \tag{1}$$

determines a well-defined vector field F_*Z on N (tangent to N if Z is tangent to M). Note: by the convention we are using "vector field" means *smooth* vector field. Therefore smoothness is something you are to *assume* about Z, but have to *prove* about F_*Z. (This problem is being assigned partly because of what I am seeing on your midterms. (i) Under a general smooth map, tangent *vectors* push forward, but general *vector fields* do not, unless the map is one-to-one—you should be able to see from the formula above why there is no good definition of $(F_*Z)_p$ if p has more than one inverse image, unless the vector field Z is very special. The situation is very similar to the one I commented on in the homework assignment posted 10/20/04. (ii) While the principle is generally true that smooth operations applied to smooth gadgets yield smooth gadgets, it is something that needs to be proven each time you invent a new gadget. At this stage of your learning of differential geometry, you should not be asserting that the object F_*Z defined above is "obviously" smooth. Only once you have had enough experience with the way to prove that F_*Z is smooth do you have a right to assert that it's obvious, or to completely ignore the issue.)

(c) Let $F : \mathbf{R}^3 \to \mathbf{R}^3$ be an *isometry* and again let $N = F(M)$; since all isometries of $\mathbf{R}^3$ are diffeomeorphisms, the results of parts (a) and (b) apply. Show that for any tangent vector field X on M and any (not necessarily tangent) vector field Z along M we have

$$F_*(\nabla_X Z) = \nabla_{F_*(X)}(F_*(Z)). \tag{2}$$

(d) Hypotheses and notation as in (c). Show that if U is a unit normal vector field along M, then F_*U is a unit normal vector field along N.

(e) Hypotheses and notation as in (c). Assume that M is oriented, let U be the corresponding unit normal vector field along M, and let $\bar{U} = F_*U$. Let S and h be, respectively, the shape operator and second fundamental form associated with the given orientation of M, and let $\bar{S}$ and $\bar{h}$ be the shape operator and second fundamental form associated with the orientation of N determined by $\bar{U}$. Show that for all tangent vector fields X, Y on M, we have

$$\bar{S}(F_*X) = F_*(S(X)) \tag{3}$$

and

$$(\bar{h}(F_*X, F_*Y)) \circ F = h(X, Y). \tag{4}$$

(Note that the assertion that (4) holds for all X, Y is identical to the assertion "$F^*\bar{h} = h$").

Equation (4) justifies the assertion made in class that if we translate a surface M so that a given point p gets taken to the origin, then rotate that translated surface to obtain a surface N whose tangent plane at the origin is the xy plane, then the relation between the second fundamental form h_p and the geometry of M near p is the same as the relation between the second fundamental form $\bar{h}_{\mathbf{0}}$ and the geometry of N near $\mathbf{0}$. Thus the second fundamental form always gives the quadratic approximation to a function $f : (O \subset T_pM) \to (T_pM)^\perp$ whose graph is the corresponding open neighborhood of p in M.

(f) Hypotheses and notation as in (e). Let $K, \bar{K}$ be the Gaussian curvature functions of M, N respectively, and let $H, \bar{H}$ be the mean curvature functions determined by the choices $U, \bar{U}$ of unit normals. Similarly let $k_1, k_2, \bar{k}_1, \bar{k}_2$ be the principal-curvature functions defined by taking $k_1(p)$ to be the larger of the two principal curvatures at each $p \in M$ and k_2 to be the smaller, etc. for $\bar{k}_1, \bar{k}_2$. Show that $\bar{K} \circ F = K$ and $\bar{H} \circ F = H$, and that $\bar{k}_i \circ F = k_i$ for $i = 1, 2$ (of course the first two equalities follow from the last, but you can prove the first two without ever mentioning principal curvatures). Prove also that F_* carries principal directions to principal directions.

The preceding shows that Gaussian, mean, and principal curvatures and principal directions are *geometric invariants* in the sense of being objects that are preserved by isometries of $\mathbf{R}^3$.