MTG 6256, Fall 2004: Non-book Problem 7

In this problem, which relies on the results of non-book problem 6(f), you will see that the types of symmetry a surface can have are quite restricted.

An isometry F of $\mathbf{R}^3$ is said to have *finite order* if for some positive integer n, F composed with itself n times is the identity; the smallest such n is then called the *order of* F. It can be shown that if F has finite order then F is conjugate to an element of the orthogonal group; i.e. $F = \tau \circ R \circ \tau^{-1}$ for some translation τ and orthogonal transformation R.

(a) Let M be a surface in $\mathbf{R}^3$, $p \in M$, and let $F : \mathbf{R}^3 \to \mathbf{R}^3$ be an isometry of finite order such that $F(M) = M$ and $F(p) = p$. (The picture to have in mind here is the case where p is the origin, T_pM is the xy plane, and F is either counterclockwise rotation about the z-axis by $2\pi/n$ for some n, or the composition of such a rotation with a reflection. The assumption that $F(M) = M$ says, in the case of a pure rotation, that M has n-fold rotational symmetry.) Prove that if the order of F is not 1,2, or 4, then p is an umbilic point of M.

(Thus, for example, a surface cannot have 3-fold or 5-fold rotational symmetry about an axis through a fixed-point p unless p is umbilic.)

(b) Hypotheses and notation as in (a). As an instant corollary of (a), deduce that if the order of F is not 1,2, or 4, then the Gaussian curvature of M at p is ≥ 0.

(c) Hypotheses and notation as in (a), but additionally assume that M has an asymptotic direction at p (defined on p. 233 of O'Neill). Prove that if F has finite order $n > 2$ then (i) if $n \neq 4$ then p is a planar point of M (defined on p. 211) and (ii) if $n = 4$ then the mean curvature of M at p is 0 (a well-defined condition even if M is not orientable). An example of case (ii) is the hyperbolic paraboloid $z = x^2 - y^2$.

(d) Hypotheses and notation as in (c). As an instant corollary of (c), deduce that if F has finite order exceeding 2, then the mean curvature of M at p is 0.

(e) Replace the hypothesis above that F fixes a point p of M by the hypothesis that the origin $\mathbf{0}$ is contained in M. Replace the hypothesis that F has finite order with the hypothesis that $F = \tau \circ R$, where τ is a translation and R is an orthogonal transformation of finite order. Show that all the conclusions of (a)–(d) still hold with p replaced by $\mathbf{0}$.

(f) Of course, most surfaces do not contain the origin. Figure out how to modify the hypotheses in (e), as minimally as possible, so as to obtain conclusions as strong as those in (e) without assuming that $\mathbf{0} \in M$.