Differential Geometry—MTG 6256—Fall 2012
Problem Set 2

1. *Covering spaces.* A *covering space* of topological space X is a pair $(\widetilde{X}, \pi)$, where $\widetilde{X}$ is a topological space and π is a continuous surjective map with the following property: for each $p \in X$, there is an open neighborhood U of p such that $\pi^{-1}(U)$ is a union of disjoint open sets in $\widetilde{X}$, each of which is mapped homeomorphically by π to U. (Surjectivity is automatic if "union" is replaced by "non-empty unuion".) The map π is called the *projection* or the *covering map.*

In both parts below, assume that $(\widetilde{M}, \pi)$ is a covering space of M.

(a) Suppose that M is a manifold of dimension n. Show that an atlas on M gives rise to an n-dimensional atlas on $\widetilde{M}$, hence that $\widetilde{M}$ naturally inherits the structure of a smooth n-dimensional manifold. (For this reason we usually refer to $\widetilde{M}$ or $(\widetilde{M}, \pi)$ as a covering *manifold* of M, rather than just a covering *space*.)

Note: I have not given a formal definition of "naturally", but once you figure out the construction of the atlas on $\widetilde{M}$, it should be clear that "naturally" is a reasonable word to apply to your construction. In place of this terminology, I could have said that there exists a unique smooth structure (equivalently, maximal atlas) on $\widetilde{M}$ such that π is a local diffeomorphism.

(b) Suppose that $\widetilde{M}$ is a manifold of dimension n. Assume that for any two open sets $\widetilde{U}_1, \widetilde{U}_2 \subset \widetilde{M}$ for which $\pi|_{\widetilde{U}_i}$ is injective and $\pi(\widetilde{U}_1) = \pi(\widetilde{U}_2)$, the map $(\pi|_{\widetilde{U}_2})^{-1} \circ \pi|_{\widetilde{U}_1}$ is smooth. (Hence all such maps are diffeomorphisms.) Show that M naturally inherits the structure of a smooth n-dimensional manifold.

2. The *Grassmannian* or *Grassmann manifold* $G_k(\mathbf{R}^n)$ is defined to be the set of k-dimensional subspaces of $\mathbf{R}^n$. (This is a generalization of projective space; $G_1(\mathbf{R}^n) = P(\mathbf{R}^n) = \mathbf{R}P^{n-1}$. Notations for the Grassmannian vary in the literature: some people use "$G_k(\mathbf{R}^n)$ for the set of subspaces of $\mathbf{R}^n$ of *co*dimension k. The notations $G_{k,n}$ and $G_{n,k}$ are also used.)

A smooth atlas on $G_k(\mathbf{R}^n)$ can constructed as follows. Observe that given any k-plane X through the origin, any sufficiently close k-plane Y through the origin is the graph of a unique linear map $T : X \to X^\perp$, where $X^\perp$ is the orthogonal complement of X. (Here "sufficiently close" means that $Y \bigcap X^\perp = \{0\}$.) For each k-element subset $I = \{i_1, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$, let X_I be the subspace consisting of all $x \in \mathbf{R}^n$ all of whose coordinates other than those in positions $i_1, \ldots, i_k$ vanish. Let $V_I \subset \mathbf{R}^n$ be the (set-theoretic) complement of $X_I^\perp$ in $\mathbf{R}^n$.

(a) Show that $\{V_I\}$ is an open cover of $\mathbf{R}^n - \{0\}$ and determines a cover $\{U_I\}$ of $G_k(\mathbf{R}^n)$, analogously to the cover used in class for $P(\mathbf{R}^n)$.

(b) Show that there is a 1–1 correspondence ϕ_I from U_I to the set of linear maps $T : X_I \to X_I^\perp$. Hence U_I is in 1–1 correspondence with the set of $(n-k) \times k$ matrices, hence with $\mathbf{R}^{k(n-k)}$.

(c) Show that the overlap maps $\phi_J \circ \phi_I^{-1}$ are smooth, and hence that $G_k(\mathbf{R}^n)$ is a manifold of dimension $k(n-k)$.

3. (a) Prove that if $F : M \to N$ is a smooth map of manifolds and $X \subset M$ is a submanifold, then $F|_X : X \to N$ (the restriction of F to X) is also a smooth map of manifolds.

(b) Let $F : M \to N$ is a smooth map of manifolds whose image is contained is a submanifold Y of N. Prove that F, viewed as a map $M \to Y$, is also a smooth map of manifolds.

4. If V is a vector space of dimension at least 1, the *projectivization* $P(V)$ is defined to be the set of lines through the origin in V, with a suitable topology. This applies whether V is a real or complex vector space; "line through the origin" means the set of real or complex multiples of a fixed nonzero vector accordingly as V is real or complex. Alternatively, $P(V) = (V - \{0\})/\sim$, where the equivalence relation $\sim$ is defined by $v \sim w = \Longleftrightarrow\ v = tw$ for some scalar t (real or complex, accordingly), and is topologized using the quotient topology. (You do not need to know what "quotient topology" means to do this problem, but you can read about it in the glossary.)

(a) *Complex projective space* $\mathbf{C}P^n := P(\mathbf{C}^{n+1})$ $(n \geq 0)$ is the projectivization of $\mathbf{C}^{n+1}$ (with complex multiplication used to define the equivalence relation). In class, by exhibiting an atlas, we showed that the real projective space $\mathbf{R}P^n := P(\mathbf{R}^{n+1})$ is an n-dimensional manifold. Use the complex analog of this atlas to show that $\mathbf{C}P^n$ is a manifold of dimension $2n$. (Note: the complex analog of our $\mathbf{R}P^n$ atlas will initially give you chart-maps whose images lie in $\mathbf{C}^n$, not $\mathbf{R}^{2n}$. But any *real* isomorphism from the two-dimensional *real* vector space $\mathbf{C}$ to $\mathbf{R}^2$, such as $z \mapsto (\mathrm{Re}(z), \mathrm{Im}(z))$, induces a real isomorphism $\mathbf{C}^n \to \mathbf{R}^{2n}$ for $n \geq 1$. For $n \geq 1$, compose your initial chart-maps with such an isomorphism to get chart-maps that land you in $\mathbf{R}^{2n}$. Handle the case $n = 0$ separately.)

Remark. There is such a thing as a complex manifold, and as you might guess, $\mathbf{C}P^n$ is a complex n-dimensional manifold. However, the concept is subtler than you might think, and for us "manifold" will always mean "real manifold" unless otherwise specified.

(b) Show that $\mathbf{C}P^1$, also called the *Riemann sphere*, is diffeomorphic to S^2.

(c) Show that the quotient map (or *projection*) $\pi : V - \{0\} \to P(V)$ is a smooth map of *real* manifolds in the following cases: (i) V is a finite-dimensional real vector space. (ii) V is a finite-dimensional complex vector space.

(d) For $V = \mathbf{C}^{n+1} \cong_{\mathbf{R}} \mathbf{R}^{2n+2}$, let H be the restriction of the projection π to the unit sphere $S^{2n+1} \subset \mathbf{R}^{2n+2}$. Show that H is surjective and smooth. In view of (b) and the Hopf map defined in class, $H : S^{2n+1} \to \mathbf{C}P^n$ is called the *generalized Hopf* *map*. (Note: there is a reason Problem 3 was given before Problem 4.)