Differential Geometry—MTG 6256—Fall 2012
Problem Set 4

1. Let M, N, Z be manifolds, $F : M \to N, G : N \to Z$ smooth maps.
   (a) Show that if $h : Z \to \mathbf{R}$ is a function, then $(G \circ F)^*(h) = F^*(G^*(h))$.
   (b) Show that if ω is a 1-form on Z, then $(G \circ F)^*(\omega) = F^*(G^*(\omega))$.
   (c) Show that if F and G are diffeomorphisms (so that pullback of a vector field is defined), and X is a vector field on Z, then $(G \circ F)^*(X) = F^*(G^*(X))$.

The principle exemplified above is universal: for any object that (naturally) pulls back under maps, we find that $(G \circ F)^*(\cdot) = F^*(G^*(\cdot))$. Similarly, for any object that (naturally) pushes forward under maps, we find that $(G \circ F)_*(\cdot) = G_*(F_*(\cdot))$. (Of course, in such vague generality, one cannot give a proof. But if you ever find one of these principles violated, you should suspect that you've made an error.)

2. Let X be a vector field on a manifold M, with flow Φ.
   (a) Recall that if μ is either a function on M or a vector field on M, or a 1-form on M, the Lie derivative of μ by X at the point p is defined by

$$(\mathcal{L}_X \mu)|_p = \frac{d}{dt} \left((\Phi_t^* \mu)|_p \right) \Big|_{t=0} . \tag{1}$$

When μ is a function, (1) defines the Lie derivative of a 0-form by X. The same equation is used to define the Lie derivative of a k-form by X for any $k > 0$.

It is natural to ask: what if we evaluate the t-derivative at general t?

Show that if (p, t_0) is in the domain of the flow, then for all of the objects μ above we have

$$\frac{d}{dt} \left((\Phi_t^* \mu)|_p \right) \Big|_{t=t_0} = \left(\Phi_{t_0}^* (\mathcal{L}_X \mu) \right) \big|_p . \tag{2}$$

(To approach this efficiently, instead of giving several separate but similar proofs, realize that whichever type of object we are Lie-differentiating, $t \mapsto (\Phi_t^* \mu)\big|_p$ is a [parametrized] curve in a *fixed* finite-dimensional vector space V_p: either $\mathbf{R}$, T_pM, or T_p^*M.)

Remark/Reminder. Once we're clear on what (2) means, we allow ourselves to rewrite it more briefly as

$$\frac{d}{dt} \Phi_t^* \mu = \Phi_t^* (\mathcal{L}_X \mu). \tag{3}$$

However, only if the vector field X is complete (i.e. the domain of Φ is $M \times R$, which is guaranteed if M is compact) is $(\Phi_t^* \mu)_p$ defined for all (p, t).

(b) Show that for all (p, t) in the domain of the flow Φ of X, we have

$$(\Phi_t^* X)_p = X_p. \tag{4}$$

(One way to get this is to use part (a). However, (4) is also equivalent to $X_{\Phi_t(p)} = \Phi_{t*p}X_p$, which can be shown directly, without using part (a).) We usually write (4) more briefly as "$\Phi_t^* X = X$" (or write the "lower-star" version as $\Phi_{t*}X = X$), with the same understanding as above about the domain of Φ.

(c) Show that $\mathcal{L}_X$ is Leibnizian with respect to wedge product:

$$\mathcal{L}_X(\omega \wedge \eta) = (\mathcal{L}_X\omega) \wedge \eta + \omega \wedge \mathcal{L}_X\eta$$

for all differential forms ω, η on M. (Note that, in contrast to the formula for $d(\omega\wedge\eta)$, there is no "$(-1)^{\deg(\omega)}$" in front of the second term this formula.)

3. Let M, N be manifolds and $F : M \to N$ a smooth map. Vector fields $\tilde{X}$ on M, X on N are said to be F-*related* if $F_{*p}\tilde{X}_p = X_{F(p)}$ for all $p \in M$. We also sometimes say that $\tilde{X}$ *projects* to X if this relation holds, but that terminology can be misleading if F is not surjective. Note that a necessary condition for a vector field $\tilde{X}$ to be "projectable"—i.e. F-related to some vector field on N—is that for all $q \in N$ and all $p_1, p_2 \in F^{-1}(\{q\})$, we must have $F_{*p_1}\tilde{X}_{p_1} = F_{*p_2}\tilde{X}_{p_2}$. Most vector fields on M will not meet this consistency condition.

Suppose that $M, N, F, \tilde{X}$, and X are as above, with $\tilde{X}$ F-related to X. Let $\tilde{\Phi}$ and Φ be the flows of $\tilde{X}$ and X respectively, with their maximal domains.

(a) Show that if $(p, t) \in \text{domain}(\tilde{\Phi})$ then $(F(p), t) \in \text{domain}(\Phi)$ (so that if $\tilde{\Phi}_t(p)$ is defined, then so is $\Phi_t(F(p))$), and that $F \circ \tilde{\Phi}_t = \Phi_t \circ F$ on domain$(\tilde{\Phi})$.

(b) Show that if $\tilde{Y}$ is another vector field on M, and is F-related to a vector field Y on N, then $[\tilde{X}, \tilde{Y}]$ is F-related to $[X, Y]$. (Use the fact that $[X, Y] = \mathcal{L}_X Y$.)

4. **(Beginning of set-up.)** For a vector space V, we have defined the exterior powers $\bigwedge^k(V^*)$ for $k \geq 0$, and have used this to define $\Omega^k(M)$, the space of k-forms on a manifold M. For certain purposes, it is convenient to extend these definitions to the case $k = -1$, setting $\bigwedge^{-1}(V^*) = \{0\} \subset \mathbf{R} = \bigwedge^0(V)$ and $\Omega^{-1}(M) = \{\text{the constant function } 0\} = \{\text{zero} - \text{elementof } \Omega^0(M)\} \subset \Omega^0(M)$.

For a vector space V, an element $v \in V$, and $k \geq 0$, we define a linear map $\iota_v : \bigwedge^k(V^*) \to \bigwedge^{k-1}(V^*)$ as follows. For $k = 0$, ι_v is the unique map $\bigwedge^0(V^*) \to \bigwedge^{-1}(V)^*$, namely the zero map. For $k > 0$, we define ι_v by

$$(\iota_v\alpha)(w_1, \ldots, w_{k-1}) = \alpha(v, w_1, \ldots, w_{k-1}) \;\; \forall w_1, \ldots, \omega_{k-1} \in V.$$

The map $\iota_v : \bigwedge^k(V^*) \to \bigwedge^{k-1}(V^*)$ $(k \geq 0)$ is called *contraction* or *interior product* with v. (One of the conveniences of having included the case $k = 0$ is that, by direct sum, we can regard ι_v as a single linear map $\bigwedge^\star(V^*) \to \bigwedge^\star(V^*)$.)

For a manifold M, a vector field X on M, and $k \geq 0$, we apply the definition above pointwise (with $V = T_p^*M, p \in M$), yielding a linear map $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$. Thus, for $k \geq 1$,

$$(\iota_X\omega)(Y_1,\ldots,Y_{k-1}) = \omega(X,Y_1,\ldots,Y_{k-1}) \tag{5}$$

for all vector fields $Y_1,\ldots,Y_{k-1}$ on M; for $f\in\Omega^0(M)$, $\iota_X f$ is the constant function 0. **(End of set-up.)**

Let M be a manifold. Below, "vector field" and "differential form" mean "vector field on M " and "differential form on M", respectivly.

(a) Let $\omega\in\Omega^k(M)$, $k\geq 0$. Find a simple relation between $\iota_X(\iota_Y\omega)$ and $\iota_Y(\iota_X\omega)$.

(b) Let ω,η be differential forms, with $\omega\in\Omega^k(M)$. Show that

$$\iota_X(\omega\wedge\eta) = (\iota_X\omega)\wedge\eta + (-1)^k\omega\wedge\iota_X\eta. \tag{6}$$

(c) Using the definition of Lie derivative, show that for $k\geq 1$, all k-forms ω, and all vector fields $X,Y_1,\ldots,Y_k$,

$$\begin{aligned} X\left(\omega(Y_1,Y_2,\ldots Y_k)\right) &= (\mathcal{L}_X\omega)(Y_1,...,Y_k) + \omega(\mathcal{L}_XY_1,Y_2,\ldots,Y_k) \\ &\quad +\omega(Y_1,\mathcal{L}_XY_2,\ldots,Y_k) + \cdots + \omega(Y_1,Y_2,\ldots,\mathcal{L}_XY_k). \end{aligned} \tag{7}$$

(d) Use the result of part (c) to show that Lie derivative by a vector field X is "Leibnizian with respect to contraction": for $k\geq 0$, all k-forms ω, and all vector fields X,Y,

$$\mathcal{L}_X(\iota_Y\omega) = \iota_{\mathcal{L}_XY}\omega + \iota_Y\mathcal{L}_X\omega.$$

(e) Show that for any vector field X and any differential form ω,

$$\iota_Xd\omega + d(\iota_X\omega) = \mathcal{L}_X\omega. \tag{8}$$

(f) Show that d commutes with Lie derivative by any vector field: for all vector fields X and differential forms ω,

$$d(\mathcal{L}_X\omega) = \mathcal{L}_Xd\omega.$$