Vector-space structure on T_pM (notes for portion of 10/11/17 lecture)

Context and notation as in 10/11/17 lecture.

Let $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ be two charts of M containing p. Let $p_\alpha = \varphi_\alpha(p), p_\beta = \varphi_\beta(p)$. Recall that, for each $q \in \mathbf{R}^n$, we have already defined a vector-space structure on $T_q\mathbf{R}^n$ and a canonical isomorphism $\iota_q : T_q\mathbf{R}^n \to \mathbf{R}^n$ given by $[\bar\gamma] \mapsto \gamma'(0)$. Below, we use the abbreviations $\iota_\alpha := \iota_{p_\alpha}, \iota_\beta := \iota_{p_\beta}$.

We have defined a bijective map $\varphi_{\alpha*p} : T_pM \to T_{p_\alpha}\mathbf{R}^n, [\gamma] \mapsto [\varphi_\alpha \circ \gamma]$, etc. for β. We have also observed that $(\varphi_{\alpha*p})^{-1}$ is the map $[\bar\gamma] \mapsto [\varphi_\alpha^{-1} \circ \bar\gamma]$.

Claim: The map $h_{\beta\alpha} := \varphi_{\beta*p} \circ (\varphi_{\alpha*p})^{-1} : T_{p_\alpha}\mathbf{R}^n \to T_{p_\beta}\mathbf{R}^n$ is linear.

Proof: Let $J = J_{\varphi_\beta \circ \varphi_\alpha^{-1}}(\varphi_\alpha(p))$, and let $g_{\beta\alpha} : \mathbf{R}^n \to \mathbf{R}^n$ be the linear map $v \mapsto Jv$. For every curve $\bar\gamma$ based at p_α, we have

$$h_{\beta\alpha}([\bar\gamma]) = \varphi_{\beta*p}\left((\varphi_{\alpha*p})^{-1}([\bar\gamma])\right) = \varphi_{\beta*p}\left([\varphi_\alpha^{-1} \circ \bar\gamma]\right) = [\varphi_\beta \circ \varphi_\alpha^{-1} \circ \bar\gamma],$$

and thus

$$\iota_\beta\left(h_{\beta\alpha}([\bar\gamma])\right) = (\varphi_\beta \circ \varphi_\alpha^{-1} \circ \bar\gamma)'(0) = J\bar\gamma'(0) = J\iota_\alpha[\bar\gamma].$$

Hence $\iota_\beta \circ h_{\beta\alpha} = g_{\beta\alpha} \circ \iota_\alpha$, so $h_{\beta\alpha} = \iota_\beta^{-1} \circ g_{\beta\alpha} \circ \iota_\alpha$, a composition of three linear maps. ■

Using the vector-space operations we've previously defined on tangent spaces of $\mathbf{R}^n$, we define vector-space operations on T_pM induced by the chart $(U_\alpha, \varphi_\alpha)$ by setting

$$v \ +_\alpha \ w = (\varphi_{\alpha*p})^{-1}(\varphi_{\alpha*p}v + \varphi_{\alpha*p}w), \quad c \ \cdot_\alpha \ v = (\varphi_{\alpha*p})^{-1}(c\,\varphi_{\alpha*p}v).$$

for all $v, w \in T_pM$ and $c \in \mathbf{R}$. We define vector-space operations on T_pM induced by the chart (U_β, φ_β) analogously. Using the linearity of $h_{\beta\alpha}$ shown above, we have

$$\begin{aligned} v \ +_\alpha \ w &= (\varphi_{\beta*p})^{-1} \circ \varphi_{\beta*p} \circ (\varphi_{\alpha*p})^{-1}(\varphi_{\alpha*p}v + \varphi_{\alpha*p}w) \\ &= \left((\varphi_{\beta*p})^{-1} \circ h_{\beta\alpha}\right)(\varphi_{\alpha*p}v + \varphi_{\alpha*p}w) \\ &= (\varphi_{\beta*p})^{-1}\left(h_{\beta\alpha}(\varphi_{\alpha*p}v) + h_{\beta_\alpha}(\varphi_{\alpha*p}w)\right) \\ &= (\varphi_{\beta*p})^{-1}\left(\varphi_{\beta*p}v + \varphi_{\beta*p}w\right) \\ &= v \ +_\beta \ w. \end{aligned}$$

Similarly, $c \ \cdot_\alpha \ v = c \ \cdot_\beta \ v$.

Hence the vector-space structures on T_pM induced by any two charts are the same. Thus T_pM has a canonical vector-space structure, the one induced by any chart containing p.