Differential Geometry—MTG 6256—Fall 2017
Problem Set 2 (updated 9/23/17)
Due-date: Mon. 10/2/17

Required problems (to be handed in; due-date TBA): 2a, 3c, 4ace. In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Optional problems: All the ones that are not required.

Below:

- Whenever we refer to an atlas on an already-fixed manifold, we mean mean an atlas within the (explicitly or implicitly) already-fixed maximal atlas.
- I use the words "natural(ly)" and "canonical(ly)" without a formal, mathematically precise, universally applicable definition. Whenever one of these words comes up, it should be clear from context what it means in that context.
- "Smooth map of manifolds" means "smooth map from one manifold to another".

1. Let M be a manifold, $U \subset M$ a nonempty open set. Show that an atlas on M naturally gives rise to an atlas on U of the same dimension, hence that U inherits a manifold structure.[1]

2. *Covering spaces.* A *covering space* of topological space X is a pair $(\widetilde{X}, \pi)$, where $\widetilde{X}$ is a topological space and $\pi : \widetilde{X} \to X$ is a continuous surjective map with the following property: for each $p \in X$, there is an open neighborhood U of p such that $\pi^{-1}(U)$ is a union of disjoint open sets in $\widetilde{X}$, each of which is mapped homeomorphically by π to U. (Surjectivity is automatic if "union" is replaced by "non-empty unuion".) The map π is called the *projection* or the *covering map*.

Below, assume that M, N are manifolds and that $(\widetilde{M}, \pi)$, $(\widetilde{N}, \pi')$ are covering spaces of M, N respectively.

(a) Let $m = \dim(M)$. Show that an atlas on M gives rise to an m-dimensional atlas on $\widetilde{M}$, hence that $\widetilde{M}$ naturally inherits the structure of a smooth m-dimensional manifold. (For this reason we usually refer to $\widetilde{M}$ or $(\widetilde{M}, \pi)$ as a covering *manifold* of M, rather than just a covering *space*.)

(b) **Definition.** Let X, Y be manifolds. A map $F : X \to Y$ is a *local diffeomorphism* if F is an open map and such that every $p \in X$ has an open neighborhood U with the property that $F|_U : U \to F(U)$ is a diffeomorphism. Here, $F(U)$ is an open set since the map F is open, and naturally carries a manifold structure by problem

[1]Once we define "submanifold", an open set as above will be the most trivial example of a submanifold.

1. (Later in the course, we will be able to replace this definition by a briefer, more standard, but less self-explanatory one that is equivalent.)

Show that the natural smooth structure (equivalently, maximal atlas) on $\widetilde{M}$ in part (a) is the unique smooth structure for which π is a local diffeomorphism.

(c) Suppose that $\widetilde{M}$ is a manifold of dimension m. Assume that for any two open sets $\widetilde{U}_1, \widetilde{U}_2 \subset \widetilde{M}$ for which $\pi|_{\widetilde{U}_i}$ is injective and $\pi(\widetilde{U}_1) = \pi(\widetilde{U}_2)$, the map $(\pi|_{\widetilde{U}_2})^{-1} \circ \pi|_{\widetilde{U}_1}$ is smooth. (Hence all such maps are diffeomorphisms.) Show that M naturally inherits the structure of a smooth m-dimensional manifold.

(d) Let $F : M \to N$ be a continuous map, and let $\tilde{F}$ be a continuous map from $\widetilde{M}$ to N, from $M \to \tilde{N}$, or from $\tilde{M} \to \tilde{N}$. If the corresponding diagram below commutes, we call $\tilde{F}$ a *lift* of F. Show that for lifts of all three types, cases, if $\tilde{F}$ is a lift of F, then $\tilde{F}$ is smooth if and only if F is smooth.

Note: Given only F, a unique lift $\tilde{F} : \widetilde{M} \to N$ always exists, namely $\tilde{F} = F \circ \pi : M \to N$. The other two types of lifts do not always exist, and when they exist, they may not be unique. Given only $\tilde{F}$, we say that $\tilde{F}$ *descends* to a map $M \to N$ if there exists $F : M \to N$ of which $\tilde{F}$ is a lift. A map $\tilde{F} : M \to \tilde{N}$ descends uniquely to the map $F = \pi' \circ \tilde{F}$; in the other two cases, $\tilde{F}$ does not always descend, but when it does descend, the map to which it descends is unique.

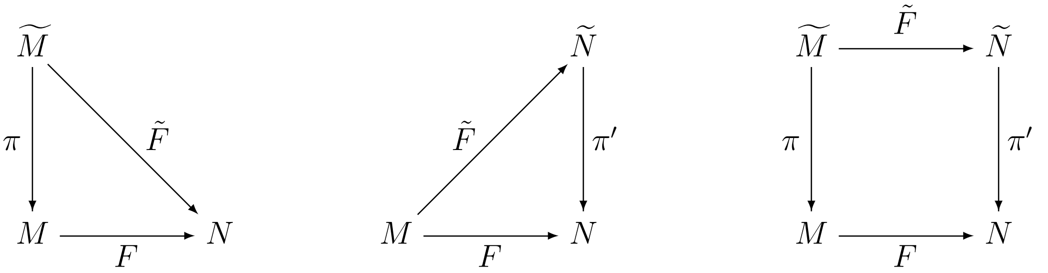

Figure 1: Diagrams for problem 2

3. Parts (a), (b), (c) of this problem have no dependence on each other; they can be done in any order.

Let $F : M \to N$ be a smooth map of manifolds.[2]

(a) Prove that if $X \subset M$ is a submanifold, then $F|_X : X \to N$ (the restriction of F to X) is also a smooth map of manifolds.

(b) Suppose that the image of F contained in a submanifold Y of N. Prove that F, viewed as a map $M \to Y$, is also a smooth map of manifolds.

(c) Define $G : M \to M \times N$ by $G(p) = (p, F(p))$. Show that G is a smooth map of manifolds, and that its image is a submanifold of $M \times N$.

[2]This is short-hand language for: "Let M, N be manifolds and let $F : M \to N$ be a smooth map."

Remark: Let us informally define "smooth subset of a manifold" to mean "submanifold". The *image* of a smooth map need be smooth (for example, consider the image of the sine function $\mathbf{R} \to \mathbf{R}$), but what you've shown above is that the *graph* of a smooth map is always smooth (as, indeed, the graph of the sine function is).

4. *Real and complex projective spaces.* If V is a vector space over a field $\mathbf{F}$, *projectivization* $P(V)$ is defined (as a set) to be the set of one-dimensional vector subspaces of V ("lines through the origin in V"). Alternatively, $P(V) = (V \setminus \{0\})/\sim$, where the equivalence relation $\sim$ is defined by $v \sim w \iff v = tw$ for some $t \in \mathbf{F}$.

Let $n \geq 0$, and let $V = \mathbf{F}^{n+1}$, where $\mathbf{F}$ is either $\mathbf{R}$ or $\mathbf{C}$. Let $\pi : V \to P(V)$ be the quotient map, and denote $\pi(v)$ by $[v]$ whenever convenient. For $0 \leq j \leq n$ define

$$\begin{aligned} \tilde{U}_j &= \{(x^0, \ldots, x^n) \in \mathbf{F}^{n+1} \mid x^j \neq 0\} \subset V, \\ U_j &= \pi(\tilde{U}_j) \subset P(V). \end{aligned}$$

Clearly $\{\tilde{U}_j\}_{j=0}^{n+1}$ is an open cover of $V \setminus \{0\}$, so $\{U_j\}_{j=0}^{n+1}$ is an open cover of $P(V)$. In view of the equivalence relation defining $P(V)$, the maps $\tilde{\phi}_j : U_j \to \mathbf{F}^n$ defined by

$$\tilde{\phi}_j(x^0, \ldots, x^n) = \left(\frac{x^0}{x^j}, \ldots, \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, \ldots, \frac{x^n}{x^j} \right) \tag{0.1}$$

induce well-defined maps $\phi_j : U_j \to \mathbf{F}^n$,

$$\phi_j([v]) = \tilde{\phi}_j(v).$$

(In (**??**), it is understood that "$\frac{x^0}{x^j}$" is omitted if $j = 0$ and that $\frac{x^n}{x^j}$" is omitted if $j = n$. Alternative notation for the right-hand side is

$$(\frac{x^0}{x^j}, \ldots, \widehat{\frac{x^j}{x^j}}, \ldots, \frac{x^n}{x^j}),$$

where the "hat" denotes deletion of the term indexed by j.)

(a) Identify the sets $\phi_j(U_j)$ and $\phi_j(U_i \bigcap U_j)$ $(i \neq j)$ explicitly, and compute the overlap-maps $\phi_i \circ \phi_j^{-1}$. (As always, in the overlap-map expression "$\phi_i \circ \phi_j^{-1}$", it is understood that "ϕ_j" is short-hand for " $\phi_j|_{U_i \cap U_j}$".)

(b) *Real projective space.* Show that $\{(U_i, \phi_i)\}_{i=0}^n$ is a smooth, n-dimensional atlas on $\mathbf{R}P^n := P(\mathbf{R}^{n+1})$. Hence $\mathbf{R}P^n$, with the corresponding maximal atlas, is an n-dimensional manifold.

Whenever anyone speaks of $\mathbf{R}P^n$ as a manifold, it's implicit that this is the smooth structure.

(c) *Complex projective space.* Any *real* isomorphism from the two-dimensional *real* vector space $\mathbf{C}$ to $\mathbf{R}^2$, such as $z \mapsto (\mathrm{Re}(z), \mathrm{Im}(z))$, induces a real isomorphism $\mathbf{C}^n \to \mathbf{R}^{2n}$ for $n \geq 1$. By composing the chart-maps ϕ_j with such an isomorphism, we obtain maps $U_j \to \mathbf{R}^{2n}$. To avoid notational clutter, in this problem we will abuse notation slightly, allowing "ϕ_j" to stand both for our previously-defined $U_j \to \mathbf{C}^n$ and for the corresponding map $U_j \to \mathbf{R}^{2n}$.

Show that $\{(U_i, \phi_i)\}_{i=0}^n$ is a smooth, $2n$-dimensional atlas on $\mathbf{C}P^n := P(\mathbf{C}^{n+1})$. In the formula for $\phi_i \circ \phi_j^{-1}$, you may treat n-tuples of complex numbers as elements of $\mathbf{R}^{2n}$ wherever necessary. Hence $\mathbf{C}P^n$, with the corresponding maximal atlas, is a $2n$-dimensional manifold.

Whenever anyone speaks of $\mathbf{C}P^n$ as a manifold, it's implicit that this is the smooth structure.

Remark. There *is* such a thing as a complex manifold, and as you might conjecture, $\mathbf{C}P^n$ is a complex n-dimensional manifold. However, the concept is subtler than you might think, and for us, "manifold" will always mean "real manifold" unless otherwise specified.

(d) For $V = \mathbf{R}^{n+1}$ and $V = \mathbf{C}^{n+1}$, show that the topology on $P(V)$ induced by the atlases in parts (b) and (c) is the same as the quotient topology. (In case you need to review the meaning of *quotient topology*, it's in the handout "Point-Set Topology: Glossary and Review" on the class home page.)

(e) Show that $\mathbf{C}P^1$ is diffeomorphic to S^2 by explicitly exhibiting a diffeomorphism $F : \mathbf{C}P^1 \to S^2$ that maps U_0 to $S^2 \setminus \{\text{north pole}\}$, and maps U_1 to $S^2 \setminus \{\text{south pole}\}$.

Remark. $\mathbf{C}P^1$ is also called the *Riemann sphere*. As a set, $\mathbf{C}P^1 = U_0 \coprod \{[(0,1)]\}$ ("$\coprod$" means "disjoint union"). In the Riemann sphere, our set U_0 is implicitly identified with $\phi_0(U_0) = \mathbf{C}$, and the point $[(0,1)]$ is regarded as "the point at infinity". If you did part (a) correctly, you should find that both overlap maps are given by $z \mapsto \frac{1}{z}$, with domain $\mathbf{C} \setminus \{0\}$.

(f) Show that the quotient map (or *projection*) $\pi : V \setminus \{0\} \to P(V)$ is a smooth map of manifolds in the cases $V = \mathbf{R}^{n+1}$ and $V = \mathbf{C}^{n+1}$.

(g) *Hopf maps.* For $V = \mathbf{C}^{n+1} \cong_{\mathbf{R}} \mathbf{R}^{2n+2}$, let H be the restriction of the projection π to the unit sphere $S^{2n+1} \subset \mathbf{R}^{2n+2}$. (i) Show that H is surjective and smooth. (Note: there is a reason one part of problem 3 was given before this problem.) (ii) For $n = 1$, let F be the diffeomorphism you found in (d), and find an explicit formula for $H \circ F : (S^3 \subset \mathbf{C}^2) \to S^2$. There is more than one map F that works in part (d), but if you found the "most obvious" one, you should find that $H \circ F$ is what I called the Hopf map in class. The name "the Hopf map" is applied to H and to $H \circ (\text{diffeo } \mathbf{C}P^1 \to S^2)$. For $n > 1$, the maps $H : S^{2n+1} \to \mathbf{C}P^n$ are called *generalized Hopf maps.*

5. The *Grassmannian* or *Grassmann manifold* $G_k(\mathbf{R}^n)$ $(0 < k < n)$ is defined to be the set of k-dimensional subspaces of $\mathbf{R}^n$. (This is a generalization of real projective space; $G_1(\mathbf{R}^n) = P(\mathbf{R}^n) = \mathbf{R}P^{n-1}$. Notations for the Grassmannian vary in the literature: some people use the notation "$G_k(\mathbf{R}^n)$" for the set of subspaces of $\mathbf{R}^n$ of *co*dimension k. The notations $G_{k,n}$ and $G_{n,k}$ are also used.)

A smooth atlas on $G_k(\mathbf{R}^n)$ can constructed as follows. Endow $\mathbf{R}^n$ with standard inner product. Observe that given any k-plane X through the origin, any sufficiently close k-plane Y through the origin is the "orthogonal graph" of a unique linear map $T : X \to X^\perp$, where $X^\perp$ is the orthogonal complement of X. ("Sufficiently close" means that $Y \bigcap X^\perp = \{0\}$. "Orthogonal graph of T" means $\{v + T(v) \mid v \in X\}$. Since $\mathbf{R}^n = X \oplus X^\perp$, there is a natural bijection between $\mathbf{R}^n$ and $X \times X^\perp$. Composing appropriately with this bijection identifies the orthogonal graph of T with the "true" graph of T.) For each k-element subset $I = \{i_1, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$, let X_I be the subspace consisting of all $x \in \mathbf{R}^n$ all of whose coordinates other than those in positions $i_1, \ldots, i_k$ vanish. Let $V_I \subset \mathbf{R}^n$ be the (set-theoretic) complement of $X_I^\perp$ in $\mathbf{R}^n$.

(a) Show that $\{V_I\}$ is an open cover of $\mathbf{R}^n \setminus \{0\}$ and determines a cover $\{U_I\}$ of $G_k(\mathbf{R}^n)$, that, for $k = 1$, reduces to the open cover used in problem 3b (modulo replacing $\mathbf{R}^{n+1}$ with $\mathbf{R}^n$).

(b) Show that there is a 1–1 correspondence ϕ_I from U_I to $\mathrm{Hom}(X_I, X_I^\perp)$. Hence U_I is in 1–1 correspondence with the set of $(n-k) \times k$ matrices, hence with $\mathbf{R}^{k(n-k)}$.

(c) Show that the overlap maps $\phi_J \circ \phi_I^{-1}$ are smooth (this requires quite a bit more work for general k than did the $k = 1$ case in problem), and hence that $G_k(\mathbf{R}^n)$ is a manifold of dimension $k(n-k)$.