Differential Geometry—MTG 6256—Fall 2017
Problem Set 3
Due-date: Friday 10/27/17

Throughout this assignment, "manifold" means "Hausdorff manifold".

Required problems (to be handed in; due-date TBA): 1, 5abc, 8, 9. In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Optional problems: All the ones that are not required.

1. Let $\mathrm{Sym}_n(\mathbf{R}) \subset M_n(\mathbf{R}) \cong \mathbf{R}^{n^2}$ be the vector subspace consisting of $n\times n$ symmetric matrices (those A for which $A^t = A$). Define $F : M_n(\mathbf{R}) \to \mathrm{Sym}_n(\mathbf{R})$ by $F(A) = A^tA$. Let $I \in M_n(\mathbf{R})$ be the identity; note that $F^{-1}(I) = \{A \in M_n(\mathbf{R}) \mid A^tA = I\}$, which is also known as the *orthogonal group* $O(n)$. Show that I is a regular value of F, and hence that $O(n)$ is a submanifold of $M_n(\mathbf{R})$. What is the dimension of $O(n)$?

Note: $O(n)$ is not connected; it has two connected components, the set $SO(n)$ of orthogonal matrices of determinant 1, and the set of orthogonal matrices of determinant -1. This example shows that non-connected manifolds can arise naturally in important examples.

2. Let M be a manifold of dimension m, and N a nonempty subset of M. (The additional hypotheses in each part below are independent of each other; they do not continue from one part to the next.)

(a) Let M' be another manifold and let $F : M \to M'$ be a diffeomorphism. Show that N is a submanifold of M if and only if $F(N)$ is a submanifold of M'. (This can be done directly, or deduced as a consequence of problem 7 in this assignment.)

(b) Let (U, φ) be a chart of M. Show that, viewed as a map from U to $\varphi(U) \subset \mathbf{R}^n$, the map φ is a diffeomorphism.

Remark: Together, parts (a) and (b) imply the following: given any chart (U, φ) of M (not necessarily adapted to N), the set $N \bigcap U$ is a submanifold of U (hence of M) if and only if $\varphi(N \bigcap U)$ is a submanifold of $\varphi(U)$ (hence of $\mathbf{R}^m$).

(c) Show that if for every $p \in N$ there exists an M-open neighborhood U of p such that $N \bigcap U$ is a submanifold of U, then N is a submanifold of M.

3. Let M, N be manifolds, let $F : M \to N$ be a smooth map, and let $p \in M$. Show that F_{*p} is an isomorphism if and only if there exist an open neighborhoods U of p, and V of $F(p)$, such that $F|_U$ is a diffeomorphism from U to V.

4. Let M, N be manifolds, with M compact and N connected. Show that if $F : M \to N$ is a submersion, then F is surjective.

5. (a) Let M, N be manifolds *of equal dimension*, with M compact and N connected. Prove that if M can be embedded in N, then M and N are diffeomorphic. (Thus, for example, the sphere S^2 cannot be embedded in the torus, or vice-versa.)

(b) Show that the assertion in part (a) would be false if the assumption "M is compact" were removed.

(c) Show that the assertion in part (a) would be false if the assumption "N is connected" were removed.

6. Let M be a manifold, $N \subset M$ a submanifold, and $j : N \to M$ the inclusion map. Show that j is an embedding.

7. Let M, N be manifolds and let $F : N \to M$ be an embedding. Show that $F(N)$ is a submanifold of M.

8. Let $F : M \to N$ be a smooth map of manifolds, let $q \in \text{image}(F)$, and assume that q is a regular value of F. Then, by the Regular Value Theorem, $Z := F^{-1}(q)$ is a submanifold of M. Let $j : Z \to M$ be the inclusion map, and let $p \in Z$. Show that

$$j_{*p}(T_pZ) = \ker(F_{*p}).$$

(In words: the tangent space at p to the fiber through p is the kernel of derivative of F at p.)

9. Let M and N be manifolds of dimensions m and n respectively. For $p \in M$, $q \in N$, how is $T_{(p,q)}(M \times N)$ related to T_pM and T_qN?

10. Let $G_k(\mathbf{R}^n)$ be the Grassmannian of k-dimensional subspaces of $\mathbf{R}^n$, $0 \le k \le n$. (See Problem Set 2.) Define $F : G_k(\mathbf{R}^n) \to G_{n-k}(\mathbf{R}^n)$ by $X \mapsto X^\perp$. Show that F is a diffeomorphism. (Thus, for example, $G_2(\mathbf{R}^3)$ is diffeomorphic to $\mathbf{R}P^3$).

11. **Transversality.** This optional multi-part problem deals with a very important concept and tool in differential topology, and the last part of it is essential to the differential-topological definition and interpretation of the *degree* of a smooth map from one compact n-dimensional manifold to another.

Notation: Given two vector subspaces U, V of a vector space W, we define their *sum* $U + V$ to be the subspace $\{u + v \mid u \in U, v \in V\}$ (also called $\text{span}\{U, V\}$).[1]

Two submanifolds M and Z of a manifold N are said to intersect *transversely* at a point $z \in N$ if $T_zM + T_zZ = T_zN$ (more precisely, if $\iota_{*z}(T_zM) + j_{*z}(T_zZ) = T_zN$, where ι, j are the inclusion maps of M, Z, respectively, into N). If this condition is met at all points of $M \bigcap Z$ we say simply that M and Z *intersect transversely*, or have *transverse intersection*, or that *the intersection is transverse*, and write $M \pitchfork Z$.

More generally, given manifolds M, N and a submanifold $Z \subset N$, a map $F : M \to N$ is said to be *transverse to Z* if for all $(p, z) \in M \times Z$ with $F(p) = z$, we have $F_{*p}(T_pM) + T_zZ = T_zN$. Short-hand notation for "F is transverse to Z" is "$F \pitchfork Z$". We may view this as a generalization of the definition in the previous paragraph, since in the case of two submanifolds M, Z of N, the submanifolds intersect transversely

[1]Note that U and V are allowed to have nontrivial intersection. When the intersection is trivial, i.e. $U \bigcap V = \{0\}$, we say that W is the *direct* sum of U and V, and (sometimes) write $W = U \oplus V$. However, we also use the symbol "$\oplus$" for the direct sum of two arbitrary vector spaces that aren't given to us as subspaces of a third.

if and only if the inclusion map $\iota : M \to N$ is transverse to Z. (It's clear that this relation is symmetric in M, Z.) Note that in this case, $\iota^{-1}(Z) = M \bigcap Z$.

Transversality comes into play when we ask the question "Is the intersection of two submanifolds a submanifold?" The answer is no in general, but yes if the intersection is transverse. Transversality is a *sufficient*, but not necessary, condition for the intersection to be a submanifold. Some examples with $N = \mathbf{R}^3$, with coordinates x, y, z: (i) the submanifolds $Z = xy$-plane, $M = yz$-plane, intersect transversely; (ii) $Z = xy$-plane, $M = z$-axis, intersect transversely; (iii) $Z = x$-axis, $M = y$-axis, do not intersect transversely ; (iv) $Z = xy$-plane, $M = \{\text{graph of } z = x^2 - y^2\}$, do not intersect transversely (because of what happens at the origin).

(a) Let $N = \mathbf{R}^n$, $0 \le k \le n$, and view N as $\mathbf{R}^k \times \mathbf{R}^{n-k}$. (For the cases $k = 0$ and $k = n$, the convention is $\mathbf{R}^0 = \{0\}$ and we make the obvious identifications of $\{0\} \times \mathbf{R}^n$ and $\mathbf{R}^n \times \{0\}$ with $\mathbf{R}^n$.) Let Z be the k-dimensional submanifold $\mathbf{R}^k \times \{0 \in \mathbf{R}^{n-k}\}$. Prove that if M is a manifold and $F : M \to N$ is transverse to Z, then $F^{-1}(Z)$ is a submanifold of M. (*Hint*: Consider the map $G = \pi \circ F : M \to \mathbf{R}^{n-k}$, where $\pi : \mathbf{R}^k \times \mathbf{R}^{n-k} \to \mathbf{R}^{n-k}$ is projection onto the second factor.)

(b) Use the result of part (a) to prove that if M, N are arbitrary manifolds and $F : M \to N$ is transverse to a submanifold $Z \subset N$, then $F^{-1}(Z)$ is a submanifold of M. (Note that the case $Z = \{\text{point}\}$ is the Regular Value Theorem, so the theorem you're asked to prove here may be considered a generalization.) What are the dimension and codimension of $F^{-1}(Z)$?

(c) Part (b), applied to the case in which F is the inclusion map of a submanifold $M \subset N$, shows that if $M \pitchfork Z$ and $M \bigcap Z \ne \emptyset$, then $M \bigcap Z$ is a submanifold of M. For $p \in M \bigcap Z$, express $T_p(M \bigcap Z)$ in terms of T_pM and T_pZ.

(d) In the setting of part (c), $M \bigcap Z$ is also a submanifold of Z, by symmetry. It is easy to show that a submanifold of a submanifold of N is a submanifold of N, so:

- M is a submanifold of N, of a certain codimension;
- Z is a submanifold of N, of a certain codimension;
- $M \bigcap Z$ is a submanifold of M, of a certain codimension;
- $M \bigcap Z$ is a submanifold of Z, of a certain codimension; and
- $M \bigcap Z$ is a submanifold of N, of a certain codimension.

Express the last three codimensions on this list in terms of the first two. To understand what these relations are saying, after you figure out the formulas, write them out without choosing letters to represent dimensions or codimensions; i.e. using the terms "codimension of M in N", "codimension of $M \bigcap Z$ in M", etc. Try to formulate a general principle that explains (not necessarily rigorously) your findings.

(e) Independent of the earlier parts of this problem, what is a necessary and sufficient condition that a subset S of a given manifold be a zero-dimensional submanifold? (The condition should involve nothing more than point-set topology.) Apply this condition when M, Z are transversely-intersecting submanifolds of N of complementary dimensions ($\dim(M) + \dim(Z) = \dim(N)$). What do you conclude about $M \bigcap Z$ in this case? If both M and Z are compact, what stronger conclusion can you reach?