Differential Geometry—MTG 6256—Fall 2017
Problem Set 4
Due-date: 11/29/17

Required problems (to be handed in): 1 (just the "(i) $\iff$ (ii)" part), 3, 7, 9, 10a, 11. In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Optional problems: All the ones that are not required.

Before starting the problems, read the notes "Bump-functions and the locality of Leibnizian linear operators" posted on the class home page. Some of the problems require facts proven in these notes. Corollary 1.9 in the notes makes use of problem 1 below, but there is no circular reasoning.

Throughout this assignment, given a manifold M:

- $\mathcal{F}(M)$ denotes the algebra of smooth functions $M \to \mathbf{R}$.
- $\text{Leib}(M)$ denotes the space of Leibnizian linear maps $\mathcal{F}(M) \to \mathcal{F}(M)$.
- For $p \in M$,
  - $\mathcal{G}_p(M)$ denotes the algebra of germs at p of smooth real-valued functions.
  - $\mathcal{F}_p(M) = \{(f, U) \mid U \text{ is an open neighborhood of } p \text{ and } f : U \to \mathbf{R} \text{ is smooth}\}$.
  - $\text{Leib}_p(M)$ denotes the space of Leibnizian linear maps $\mathcal{F}(M) \to \mathbf{R}$.
  - $\text{Leib}_p^{\mathcal{G}}(M)$ denotes the space of Leibnizian linear maps $\mathcal{G}_p(M) \to \mathbf{R}$.

Remark: A Leibnizian linear map from one algebra (over a given field, in this case $\mathbf{R}$) to another is also called a *derivation*. The term "a derivation *on* [or *of*] an algebra" is usually reserved for a derivation from an algebra to itself.

Note that for any nonempty set S and vector space V, the set $\text{Func}(S, V)$ of *all* functions $S \to V$ inherits the structure of a vector space (via pointwise operations). It is easily seen that $\text{Leib}(M)$ and $\text{Leib}_p(M)$ are vector subspaces of $\text{Func}(\mathcal{F}(M), \mathcal{F}(M))$ and $\text{Func}(\mathcal{F}(M), \mathbf{R})$, respectively, hence are vector spaces (canonically). This is the meaning of "space" in "space of Leibnizian linear maps".

1. Let X be a "set-theoretic" vector field on a manifold M. Show that the following are equivalent:

(i) Viewed as a map $M \to TM$, X is smooth.

(ii) For every chart (U, φ), with associated local coordinates $\{x^i\}_{i=1}^n$, the functions $X^i : U \to \mathbf{R}$ defined pointwise by $X|_U = \sum_i X^i \frac{\partial}{\partial x^i}$ are smooth.

(iii) For all (nonempty) open sets $U \subset M$ and smooth functions $f : U \to \mathbf{R}$, the function $X(f) : U \to \mathbf{R}$ is smooth.

2. Let M be a manifold and let $p \in M$. Show that there is a canonical isomorphism $\mathrm{Leib}_p^{\mathcal{G}}(M) \to \mathrm{Leib}_p(M)$.

Remark. Since we have previously exhibited a canonical isomorphism $T_pM \to \mathrm{Leib}_p^{\mathcal{G}}(M)$, we therefore have a canonical isomorphism $T_pM \to \mathrm{Leib}_p(M)$. Thus, instead of regarding a vector $v \in T_pM$ as an operator on germs, or as a map $\mathcal{F}_p(M) \to \mathbf{R}$ that determines such an operator, we can regard v simply as an operator on $\mathcal{F}(M)$ (real-valued, linear, and Leibnizian, of course). This fact affords some convenience, since it is simpler to say "Let f be a smooth function $M \to \mathbf{R}$" than to say "Let U be an open neighborhood of p and let $f : U \to \mathbf{R}$ be a smooth function," or to say "Let g be a smooth germ at p and let (f, U) be a representative of g." This fact is used in many definitions and proofs involving tangent vectors and/or vector fields. However, it is still important to remember that for $v \in T_pM$ and $f \in \mathcal{F}(M)$, the value of $v(f)$ depends only on the germ of f at p.

3. Let M be a manifold, let $p \in M$, and let $v \in T_pM$. Show that there exists a vector field X on M with $X_p = v$. (In other words, every tangent vector at a point can be extended to a vector field on M.)

4. Let M be a manifold and let $\pi : TM \to M$ be the natural projection. Show in the following two ways that π is a submersion: (a) by using charts $(\tilde{U}, \tilde{\varphi})$ of TM induced by charts (U, φ) of M as discussed in class; (b) using problem 3, the Chain Rule for maps of manifolds, and simple linear algebra. (*Hint*: View a vector field X as a map $X : M \to TM$ satisfying $\pi \circ X =$ identity.) You may substitute a different method for part (a) if you've figured out another proof (different from the one in (b)).

5. Let M be a manifold. Show that every vector field X on M, viewed as a map $M \to TM$, is an embedding. (The hint given in problem 4 is useful here too.)

Remark 1. It follows that the image of X is a submanifold of TM diffeomorphic to M. Note that every manifold has a canonical vector field, namely the zero vector field. Both this vector field and the corresponding submanifold of TM are called the *zero-section*.

Remark 2. In case you know what a vector bundle is: problems 3–5 and Remark 1 generalize easily to any vector bundle.

6. Let M be a manifold, X and Y vector fields on M. Viewing X, Y as linear operators $\mathcal{F}(M) \to \mathcal{F}(M)$, let $[X, Y]$ be the commutator of X and Y (the operator $\mathcal{F}(M) \to \mathcal{F}(M)$ defined by $[X, Y](f) = X(Y(f)) - Y(X(f))$).

(a) Without using local coordinates or any facts about Lie derivatives (a topic to be covered shortly after this homework problem is being posted), show directly that $[X, Y]$ is a Leibnizian linear operator $\mathcal{F}(M) \to \mathcal{F}(M)$, hence is a vector field. (This exercise was done in class, but it wouldn't hurt you to redo it.)

(b) In the local coordinates $\{x^i\}$ given by some chart, suppose that $X = \sum_i X^i \frac{\partial}{\partial x^i}$ and $Y = \sum_i Y^i \frac{\partial}{\partial x^i}$. Compute the local-coordinate expression for $[X, Y]$. (The answer to this was given in class, but you must still derive the answer.)

7. Let X be a vector field on a manifold M and let Φ the flow of X (with its maximal domain). Prove that fixed-points of the flow—i.e. those points $p \in M$ for which $\Phi_t(p) = p$ for all $t \in \mathbf{R}$—are exactly those p at which $X_p = 0$.

8. Notation as in problem 7. An *integral curve* of X is a "Φ-orbit", i.e. a set of the form $\{\Phi_t(p)\}$ with p fixed and t varying over an open interval containing 0. Prove that multiplying X by a nonvanishing function only reparametrizes the integral curves; it does not change the underlying point-sets. More precisely, if $Y = fX$ for some nowhere-zero function f, prove that every integral curve of Y is an integral curve of X and vice-versa.

9. Let M be a manifold, X a vector field on M, and suppose that $X_p \neq 0$ at some $p \in M$. Prove that there are local coordinates $\{x^i\}$ on some open neighborhood U of p such that $X = \partial/\partial x^1$ on U. (Hint: use the flow of X, and cleverly choose a chart—you figure out how—in which "time" becomes the coordinate x^1.)

Note: "there are local coordinates $\{x^i\}$ on some open neighborhood U of p" is just another way of saying "there is a chart (U, ϕ) with $p \in U$"; it is understood that the $\{x^i\}$ are the standard coordinate functions on $\phi(U) \subset \mathbf{R}^n$, ($n = \dim(M)$) pulled back to U.

Warning. Do not forget that for $n > 1$, in order to define even a single partial derivative in $\mathbf{R}^n$ (or an open subset of $\mathbf{R}^n$), you need a *complete* set of coordinates $\{x^i\}_{i=1}^n$. In order to hold all but one variable fixed, and differentiate with respect to the other, in general you need to know *all* n coordinates of every point. For example, in $\mathbf{R}^2$ with standard coordinates (x, y), suppose we define new coordinates (u, v) by $u(x, y) = x, v(x, y) = y + x$. (Equivalently, we compose the coordinate-functions $x, y : \mathbf{R}^2 \to \mathbf{R}$ with the inverse of the diffeomorphism $(x, y) \mapsto G(x, y) = (x, y + x)$, namely the map $(u, v) \mapsto G^{-1}(u, v) = (u, v - u) =$ "$(x(u, v), y(u, v))$", where at the end we've used the Calculus 3 notation $(x(u, v), y(u, v))$ for the inverse of G.) Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = x + y = v(x, y)$. Then in Calculus 3 notation, $f(x(u, v), y(u, v)) = v$ and $\frac{\partial f}{\partial u} = 0 \neq 1 = \frac{\partial f}{\partial x}$, even though $u(p) = x(p)$ for every $p \in \mathbf{R}^2$. (In precise notation, "$\frac{\partial f}{\partial u}$" is $\frac{\partial f(G^{-1}(u,v))}{\partial u}$.)

10. Let X be a vector field on a manifold M, with flow Φ.

(a) Recall that if μ is either a real-valued function on M or a vector field on M, the Lie derivative of μ by X at the point p is defined by

$$(\mathcal{L}_X \mu)|_p = \frac{d}{dt} \left((\Phi_t^* \mu)|_p \right) \Big|_{t=0} . \tag{1.1}$$

It is natural to ask: what if we evaluate the t-derivative at more general t?

Show that if (p, t_0) is in the domain of the flow, then for both types of object μ above we have

$$\frac{d}{dt}\left((\Phi_t^* \mu)\big|_p \right)\Big|_{t=t_0} = \left(\Phi_{t_0}^*(\mathcal{L}_X \mu)\right)\big|_p . \tag{1.2}$$

(To approach this efficiently, instead of giving several separate but similar proofs, realize that whichever type of object we are Lie-differentiating, $t \mapsto (\Phi_t^*\mu)\big|_p$ is a [parametrized] curve in a *fixed* finite-dimensional vector space V_p: either $\mathbf{R}$ or T_pM.)

Remark/Reminder. Once we're clear on what (1.2) means, we allow ourselves to rewrite it more briefly as

$$\frac{d}{dt}\Phi_t^*\mu = \Phi_t^*(\mathcal{L}_X\mu). \tag{1.3}$$

However, only if the vector field X is complete (i.e. the domain of Φ is $M \times R$, which—as we have seen—is guaranteed if M is compact) is $(\Phi_t^*\mu)_p$ defined for all $(p, t) \in M \times \mathbf{R}$.

(b) Show that for all (p, t) in the domain of the flow Φ of X, we have

$$(\Phi_t^* X)_p = X_p. \tag{1.4}$$

11. Let M, N be manifolds and $F : M \to N$ a smooth map. Vector fields $\tilde{X}$ on M, X on N are said to be *F-related* if $F_{*p}\tilde{X}_p = X_{F(p)}$ for all $p \in M$. (We also sometimes say that $\tilde{X}$ *projects* to X if this relation holds, but "projects" can be misleading if F is not surjective.) A necessary condition for a vector field $\tilde{X}$ on M to be "projectable"—i.e. F-related to some vector field on N—is that for all $q \in N$ and all $p_1, p_2 \in F^{-1}(\{q\})$, we have $F_{*p_1}\tilde{X}_{p_1} = F_{*p_2}\tilde{X}_{p_2}$. If F is not one-to-one, most vector fields on M will not meet this consistency condition.

Although "being F-related" is not a symmetric relation (unless F is a diffeomorphism), if $\tilde{X}$ and X are F-related, we allow ourselves to say that each of these vector fields is F-related to the other, as long as we have stated clearly which vector field is defined on which manifold.

Suppose that $M, N, F, \tilde{X}$, and X are as above, with $\tilde{X}$ and X being F-related. Let $\tilde{\Phi}$ and Φ be the flows of $\tilde{X}$ and X respectively, with their maximal domains.

(a) Show that if $(p, t) \in \text{domain}(\tilde{\Phi})$ then $(F(p), t) \in \text{domain}(\Phi)$ (equivalently: if $\tilde{\Phi}_t(p)$ is defined, then so is $\Phi_t(F(p))$), and that $F \circ \tilde{\Phi}_t = \Phi_t \circ F$ on domain$(\tilde{\Phi}_t)$.

(b) Show that if $\tilde{Y}$ is another vector field on M, and is F-related to a vector field Y on N, then $[\tilde{X}, \tilde{Y}]$ is F-related to $[X, Y]$. (Use the fact that $[X, Y] = \mathcal{L}_X Y$.)

[ASSIGNMENT CONTINUES ON NEXT PAGE]

12. The integral curves of a nonzero vector field on a manifold M are one-dimensional submanifolds of M. Given *two* vector fields X and Y, one can ask whether their integral curves "hang together" to produce two-dimensional submanifolds (at least locally), the points of any one of which can be connected to each other by moving along piecewise-smooth curves each segment of which belongs to an integral curve of X or Y. This problem gives a sufficient condition on X and Y for them to generate a two-dimensional submanifold locally in this way. The condition is also necessary in a certain sense (see part (c)). A more general sufficient condition is given in part (d). (Note: Each part of this problem from (b) on depends on part (a). If you can't do part (a), you may assume it in order to try the later parts.)

Let M be a manifold, X and Y vector fields on M, with flows Φ and Ψ respectively. We say *the flows of X and Y commute* if for any $p \in M$ and any open intervals I, J containing 0 such that if $\Phi_t \circ \Psi_s(p)$ and $\Psi_s \circ \Phi_t(p)$ are defined for all $(t, s) \in I \times J$, then $\Phi_t \circ \Psi_s(p) = \Psi_s \circ \Phi_t(p)$ for all $(t, s) \in I \times J$.

(a) Prove that $[X, Y] \equiv 0$ iff the flows of X and Y commute. (One direction of the "iff" is much easier than the other. The hard direction is what's needed below.)

(b) (Generalization of problem 9.) Suppose that X and Y are linearly independent at a point p, and that $[X, Y] \equiv 0$. Prove that there are local coordinates $\{x^i\}$ on some open neighborhood U of p such that $X = \partial/\partial x^1$ and $Y = \partial/\partial x^2$ on U. (Hint: use the flows.)

(c) Hypotheses as in (b). Prove that there exists a two-dimensional submanifold L of M, containing p, such that at each $q \in L$ the tangent space T_qL is spanned by X_q and Y_q. Conversely (more or less), show that given any two-dimensional submanifold L of M, and any $p \in L$, there exist vector fields X, Y defined on an open neighborhood U of p in M that are linearly independent at every point of U and satisfy $[X, Y] \equiv 0$ on U. (The reason for the "more or less" is that your X and Y are required to be linearly independent not just at p, but throughout U. However, you will probably find that the same proof you'd have used to show linear independence at p works at every point of your U.)

(d) Suppose that on some open set V, the vectors X_q and Y_q are linearly independent at each $q \in V$ and that $[X, Y]_q$ lies in the span of X_q and Y_q. (Thus there are functions f and g such that $[X, Y] = fX + gY$ for some unique functions f and g. Problems 6b and 9, together with the "(i) implies (ii)" part of problem 1, can be used to show that f and g are smooth. There are other ways to prove smoothness of f and g, of course; this is just one of the quickest ways, given what we've already proven.) Prove for each $p \in V$ there exist a neighborhood U of p and locally-defined vector fields $\tilde{X}, \tilde{Y}$ with the same span as X and Y at each point of U, satisfying $[\tilde{X}, \tilde{Y}] \equiv 0$. (Hence, from part (c) these conditions on X and Y can replace the less general conditions in part (b).)