Differential Geometry—MTG 6256—Fall 2017
Problem Set 5
Due-date: 12/6/17

Required problems (to be handed in): 1, 2, 6bce. In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Required reading : The statements of all the problems and Remarks. (Consider the Remarks to be part of one of the missed lectures.)

Optional problems: All the ones that are not required.

1. Let M be a manifold and let $f, g : M \to \mathbf{R}$ be smooth functions. Show that

$$d(fg) = g\,df + f\,dg.$$

2. Let M be a manifold, $p \in M$, $\xi \in T_p^*M$. Prove that there exists a smooth function $f : M \to \mathbf{R}$ such that $f(p) = 0$ and $df|_p = \xi$.

3. Let V be an n-dimensional vector space, $0 < n < \infty$, let $\{\theta^i\}_{i=1}^n$ be a basis of V^*, and let $p \in \{1, 2, \ldots, n\}$. Show that

$$\left\{\theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_p}\right\}_{1 \le i_1 < i_2 < \cdots < i_p \le n}$$

is a basis of $\bigwedge^p(V^*)$. (I.e. fill in the details of the proof that was sketched in class.)

4. Recall that a *projection* from a vector space V to itself is a linear map $P : V \to V$ such that $P^2 = P$.

Let V be a vector space of finite dimension n, assume $0 < k \le n$, and define a linear map $Q : V^{\otimes k} := \overbrace{V \otimes \ldots \otimes V}^{k \text{ factors}} \to \bigwedge^k(V)$ by setting

$$Q(v_1 \otimes \ldots \otimes v_k) = v_1 \wedge \cdots \wedge v_k$$

and extending linearly. Show that $Q = c(k)P$ for some projection $P : V^{\otimes k} \to \bigwedge^k(V) \subset V^{\otimes k}$ and scalar $c(k)$, and give the value of $c(k)$.

5. Let V be a finite-dimensional vector space. For all $j, k \ge 0$, we have defined the wedge-product map $\bigwedge^j V^* \times \bigwedge^k V^* \to \bigwedge^{j+k} V^*$, $(\omega, \eta) \mapsto \omega \wedge \eta$. Thus we have actually defined a *collection* of wedge-product maps, indexed by pairs (j, k) of non-negative integers.

(a) Show that this collection of maps is associative, in the following sense: for all $j, k, l \ge 0$ and $\omega \in \bigwedge^j V^*, \eta \in \bigwedge^k V^*, \xi \in \bigwedge^l V^*$,

$$(\alpha \wedge \beta) \wedge \xi = \alpha \wedge (\beta \wedge \xi). \tag{1.1}$$

(b) **Set-up**: Careful wording had to be used in part (a), because, by definition, the only operations that can be *associative* are binary operations *on a single set S*, i.e.x maps $S \times S \to S$. What equation (1.1) actually says, in temporary, self-explanatory notation that we'll use for the rest of this problem, is

$$(\alpha \wedge_{j,\,k} \beta) \wedge_{j+k,\,l} \xi = \alpha \wedge_{j,\,k+l} (\beta \wedge_{k,\,l} \xi). \tag{1.2}$$

There are two ways we can modify our definition of wedge-product so that "$\wedge$" becomes a true associative operation on some set S. The naive way is to take the underlying set for the operation to be $S = \coprod_k \bigwedge^k V^*$. Then, for any $\omega, \eta \in S$, we have $\omega \in \bigwedge^j V^*, \eta \in \bigwedge^k V^*$ for some unique j, k, and we can make definition $\omega \wedge \eta = \omega \wedge_{j,\,k} \eta$. (The last equation was used implicitly when we defined wedge-product in class.) With this definition, $\wedge$ becomes a map $S \times S \to S$, and part (a) shows that this binary operation is associative.

A more elegant and useful (if initially less intuitive) solution Instead of defining wedge-product as an operation on the *disjoint union* of the vector spaces $\bigwedge^k V^*$, we define it as an operation on the *direct sum* of these vector spaces. Specifically, define $\bigwedge^*(V*) = \bigoplus_{k \geq 0} \bigwedge^k V^*$. (The star in "$\bigwedge^*$" has nothing to do with pullback or dualization; it's just a placeholder for the degrees in the direct summands.) Rather than writing an element of the direct sum in the form $(\omega_0, \omega_1, \omega_2, \dots)$, with $\omega_k \in \bigwedge^k V^*$, it is convenient to use the canonical identification of $\bigwedge^k V^*$ with a subspace of the direct sum (namely $\{0\} \times \{0\} \times \cdots \times \{0\} \times \bigwedge^k V^* \times \{0\} \times \{0\}$), allowing us to write $(\omega_0, \omega_1, \omega_2, \dots)$ as $\sum_k \omega_k := \omega_0 + \omega_1 + \omega_2 + \dots$ (well-defined since there are only finitely many nonzero terms in the sum). We then define

$$\left(\sum_k \omega_k\right) \wedge \left(\sum_k \eta_k\right) = \sum_{k,\,l} \omega_k \wedge_{k,\,l} \eta_l \tag{1.3}$$

Problem: Show that the map $\wedge : \bigwedge^* V^* \times \bigwedge^* V^* \to \bigwedge^* V^*$ defined by (1.3) is bilinear and associative.

Remark. For both the "naive" and "elegant" ways of defining approach to defining wedge-product as a binary operation $\wedge$ on some set, the restriction of $\wedge$ to $\bigwedge^j V^* \times \bigwedge^k V^*$ is precisely the map $\wedge_{j,\,k}$, which allows us to drop the subscripts (and write (1.2) as (1.1)) without abusing notation. However, with the naive definition, "bilinear" is meaningless, since the disjoint union of vector spaces is not a vector space. The property of bilinearity is an important feature of wedge-product. Although each of the maps $\wedge_{j,k}$ is bilinear, if we want to have a true associative operation "$\wedge$" that is also bilinear, we must use the second approach.

We have already seen the second approach used in our definition of the tensor algebra of a vector space. The idea is very general and important, and is encapsulated in the concept of a **Z**-*graded algebra*.

A **Z**–*graded vector space* is a vector space of the form $W_* := \bigoplus_{k \in \mathbf{Z}} W_k$, where $\{W_k\}_{k \in \mathbf{Z}}$ is a collection of vector spaces indexed ("graded") by **Z**. For simplicity, canonically identify W_k with its image in the direct sum. A **Z**-*graded algebra* is a

Z-graded vector space W_* equipped with a bilinear map $\star : W_* \times W_* \to W_*$ that, for each index-pair (j,k), maps $W_j \times W_k$ into W_{j+k}.

More generally, given all the data above but with the index-set **Z** replaced by any nonempty subset A of **Z** closed under addition (setting $W_* = \bigoplus_{k\in A} W_k$), we can define a **Z**-graded algebra $(\tilde{W}_*, \tilde{\star})$ by (i) defining $\tilde{W}_k = W_k$ for $k \in A$, and $\{0\}$ for $k \notin A$, (ii) setting $\tilde{W}_* = \bigoplus_{k\in \mathbf{Z}} \tilde{W}_k$ (which we may canonically identify with $W_* \oplus W'_*$, where $W'_* = \bigoplus_{k\notin A} \tilde{W}_k \cong \{0\}$), and (iii) defining $w \tilde{\star} v = w \star v$ if $w, v \in W_*$, and $w \tilde{\star} v = 0$ otherwise. The inclusion map of W_* in $\tilde{W}_*$ is a graded-algebra isomorphism: an isomorphism of **Z**-graded algebras that preserves grading. Since this identification of $(W_*, \star)$ with a **Z**-graded algebra is canonical, we allow ourselves to call $(W_*, \star)$ itself a **Z**-graded algebra. In particular, we do this often when A consists of the non-negative integers.

If the data we start with are just a collection $\{W_k\}_{k\in A\subset \mathbf{Z}}$ of vector spaces (with A nonempty and closed under addition) and a collection of bilinear maps $\star_{j,k} : W_j \times W_k \to W_{j+k}$ (one map for each index-pair (j,k)), we canonically construct a **Z**-graded algebra by setting $W_* = \bigoplus_{k\in A} W_k$ and defining $\star : W_* \times W_* \to W_*$ by $\left(\sum_j w_j\right) \star \left(\sum_j v_j\right) = \sum_{j,k} w_j \star_{j,k} v_k$. (Remember that by definition of "direct sum of an arbitrary collection of vector spaces", there are only finitely many nonzero terms in $\sum_j w_j$ and $\sum_j v_j$, so the sums in this equation are well-defined.) It is easily seen that $\star$ is bilinear and maps $W_j \times W_k$ into W_{j+k} for all $j,k \in A$. Hence $(W_*, \star)$ is a **Z**-graded algebra. If the collection of maps $\star_{j,k}$ is "associative in the sense of problem 5a", then $\star$ is (truly) associative, and $(W_*, \star)$ is an associative algebra.

All of the above works with the index-set **Z** replaced by any abelian group G, so we may speak of G-graded algebras. The grading-groups that arise most often are **Z** and $\mathbf{Z}_2$.

6. Let M, N, Z be manifolds. Recall that given a smooth map $F : M \to N$ and a k-form ω on N, with $k > 0$, the *pullback of ω by F* is the k-form $F^*\omega$ on M defined by

$$(F^*\omega)_p(v_1, \ldots, v_k) = \omega_{F(p)}(F_{*p}\, v_1, \ldots, F_{*p}\, v_k) \ \ \forall p \in M \text{ and } v_1, \ldots, v_k \in T_pM. \quad (1.4)$$

(a) Let $F : M \to N$ be a smooth map and let $k \geq 0$. Show that the map $\Omega^k(N) \to \Omega^k(M)$ given by $\omega \mapsto F^*\omega$ is linear.

(b) Let $F : M \to N$ be a smooth map and let ω, η be differential forms on N. Show that $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$. (Do not forget the case in which the degree of ω or η is zero.)

(c) Let $F : M \to N$ and $G : N \to Z$ be smooth maps, and let ω be a differential form (of arbitrary degree) on Z. Show that $(G \circ F)^*\omega = F^*(G^*\omega)$.

(d) Show that if $F : M \to N$ is a diffeomorphism, then the linear map $F^* : \Omega^k(N) \to \Omega^k(M)$ is invertible for each k, with inverse given by $(F^*)^{-1}\eta = (F^{-1})^*\eta$.

(e) Show that if $F : M \to N$ is a diffeomorphism, $\omega \in \Omega^1(N)$, and X is a vector field on N, then

$$F^*(\langle \omega, X\rangle) = \langle F^*\omega, F^*X\rangle. \tag{1.5}$$

(In (1.5), $\langle \omega, X\rangle$ is the function $p \mapsto \langle \omega_p, X_p\rangle$; both sides of the equation are real-valued functions on M.) Where is the assumption that F is a diffeomorphism used?

Remark 1. Using the same idea as in problem 5 and the subsequent Remark, we define $\Omega^*(M) = \bigoplus_{k\geq 0} \Omega^k(M)$, and use the collection of wedge-product maps $\Omega^j(M) \times \Omega^k(M) \to \Omega^{j+k}(M)$ to define wedge-product as a bilinear map $\Omega^*(M) \times \Omega^*(M) \to \Omega^*(M)$. The space $\Omega^*(M)$, equipped with this wedge-product, is an associative $\mathbf{Z}$-graded algebra. Parts (a) and (b), combined, are therefore equivalent to the simple statement that, for any smooth map $F : M \to N$, the map $F^* : \Omega^*(N) \to \Omega^*(M)$ is a graded-algebra isomorphism.

Remark 2. Given finite-dimensional vector spaces V, W, and a linear map $L : V \to W$, the *natural adjoint* of L (or *dual map*) is the linear map $L^* : W^* \to V^*$ defined by setting $L^*(\xi) = \xi \circ L$ for all $\xi \in W^*$. The notation L^* is consistent with our notation "pullback of a real-valued function by a map": given a map $\xi : W \to \mathbf{R}$, and a map $L : V \to W$, we pull ξ back to a function on V simply by composing on the right with L. However, there is more going on.

Using composition, we *could* similarly pull back *any* function on W by *any* map $V \to W$, even if V and W were merely *sets*. But here they are vector spaces, and the map L is linear. Still, even knowing that $L : V \to W$ is a linear map between vector spaces, we could pull back *any* map $f : W \to \mathbf{R}$ (not necessarily linear) by L, and write L^*f for the pulled-back map $f \circ L : V \to \mathbf{R}$. We could then view "$L^*$" as a map from $\{$all functions $W \to \mathbf{R}\}$ to $\{$all functions $V \to \mathbf{R}\}$, and this "grand" map L^* would even be linear.[1] But this is not customarily what the notation L^* means when L is linear.

Rather, in this context—where the essential ingredients are that V, W are vector spaces (rather than that they are manifolds, or arbitrary sets) and that L is linear—we restrict attention to pulling back *linear* maps $\xi : W \to \mathbf{R}$ to maps $\xi \circ L : V \to \mathbf{R}$. The resulting maps $\xi \circ L$ are themselves linear, and the map $W^* \to V^*$ given by $\xi \mapsto \xi \circ L$ is linear, so the "grand" pullback operation, by L, from $\{$*all* functions $W \to \mathbf{R}\}$ to $\{$*all* functions $V \to \mathbf{R}\}$, restricts to a linear map from $\{$*linear* functions $W \to \mathbf{R}\}$ to $\{$*linear* functions $V \to \mathbf{R}\}$. We reserve the notation L^* for this restricted pullback operation, a linear map $W^* \to V^*$.

Observe that using dual-pairing notation we could write the definition of L^* as:

$$\langle L^*\xi, v\rangle = \langle \xi, Lv\rangle \quad \forall \xi \in W^*,\ v \in V.$$

[1] (1) Remember that for any nonempty set S and any vector space Z, the set $\mathrm{Maps}(S, Z)$ of all functions from S to Z has a natural vector-space structure, induced by pointwise operations ($(f + g)(p) = f(p) + g(p)$, etc.). Hence *linearity* of a map from one such map-space to another is well-defined. (2) Note that the *dimension* of $\mathrm{Maps}(S, \mathbf{R})$ is the *cardinality* of S. Thus $\mathrm{Maps}(V, \mathbf{R})$ has uncountable dimension if V is a nonzero vector space.

Hence, given manifolds M, N, a smooth map $F : M \to N$, and a point $p \in N$, the natural adjoint of the linear map $F_{*p} : T_pM \to T_{F(p)}N$ is the linear map $(F_{*p})^* : T^*_{F(p)}N \to T^*_pN$ defined by setting

$$\langle (F_{*p})^*\xi, v\rangle = \langle \xi, F_{*p}\, v\rangle \quad \forall \xi \in T^*_{F(p)}N,\ v \in T_pM. \tag{1.6}$$

(So, just as tangent vectors naturally push forward under general smooth maps, *co*vectors naturally pull back.)

The notation "$(F_{*p})^*$" is rarely used, since it is somewhat bewildering to look at: the upper-star is denoting a pullback *by the linear map* F_{*p}, which already contains a lower-star denoting a push-forward induced *by the map* F. Let us temporarily use the notation F^{*p} for $(F_{*p})^*$ (*temporarily* because the notation is unconventional.) Then the equation in (1.6) can alternatively be written as

$$(F^{*p}(\xi))(v) = \xi(F_{*p}\, v).$$

Thus, when $k = 1$, (1.4) says that $(F^*\omega)_p = F^{*p}(\omega_{F(p)})$. In other words, we pull back 1-forms by pulling back their values (covectors) pointwise, using the natural adjoints of the derivatives F_{*p}.

7. Let M be a manifold. Define $\Omega_0(M) = \bigoplus_{k\ \mathrm{even}} \Omega^k(M),\ \Omega_1(M) = \bigoplus_{k\ \mathrm{odd}} \Omega^k(M)$. Here we are regarding the subscripts 0 and 1 as the elements of the group $\mathbf{Z}_2$, indexing two vector spaces. This gives $\Omega^*(M)$ the structure of a $\mathbf{Z}_2$-graded vector space. Show that $\Omega^*(M)$, equipped with wedge-product, is a $\mathbf{Z}_2$-graded *algebra*.