# Submanifolds

**Examples** (before def.): submanifolds of $\mathbb{R}^n$

① $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ ; $\mathbb{R}^k \times \underset{\mathbb{R}^{n-k}}{\underset{\cap}{\{0\}}} \subset \mathbb{R}^n$

② $S^{n-1} \subset \mathbb{R}^n$

③ open set $\subset \mathbb{R}^n$

④ "nice" level-set of smooth function

$x^2 - y^2 = 1$ is "nice"

$x^2 - y^2 = 0$ is "not nice" (letter X)

**Examples in other manifolds**

⑤ Equator $\subset$ sphere

⑥ "Circle of rational slope" in $S^1 \times S^1$

**Non-example**

⑦ "Line of irrational slope" in $S^1 \times S^1$

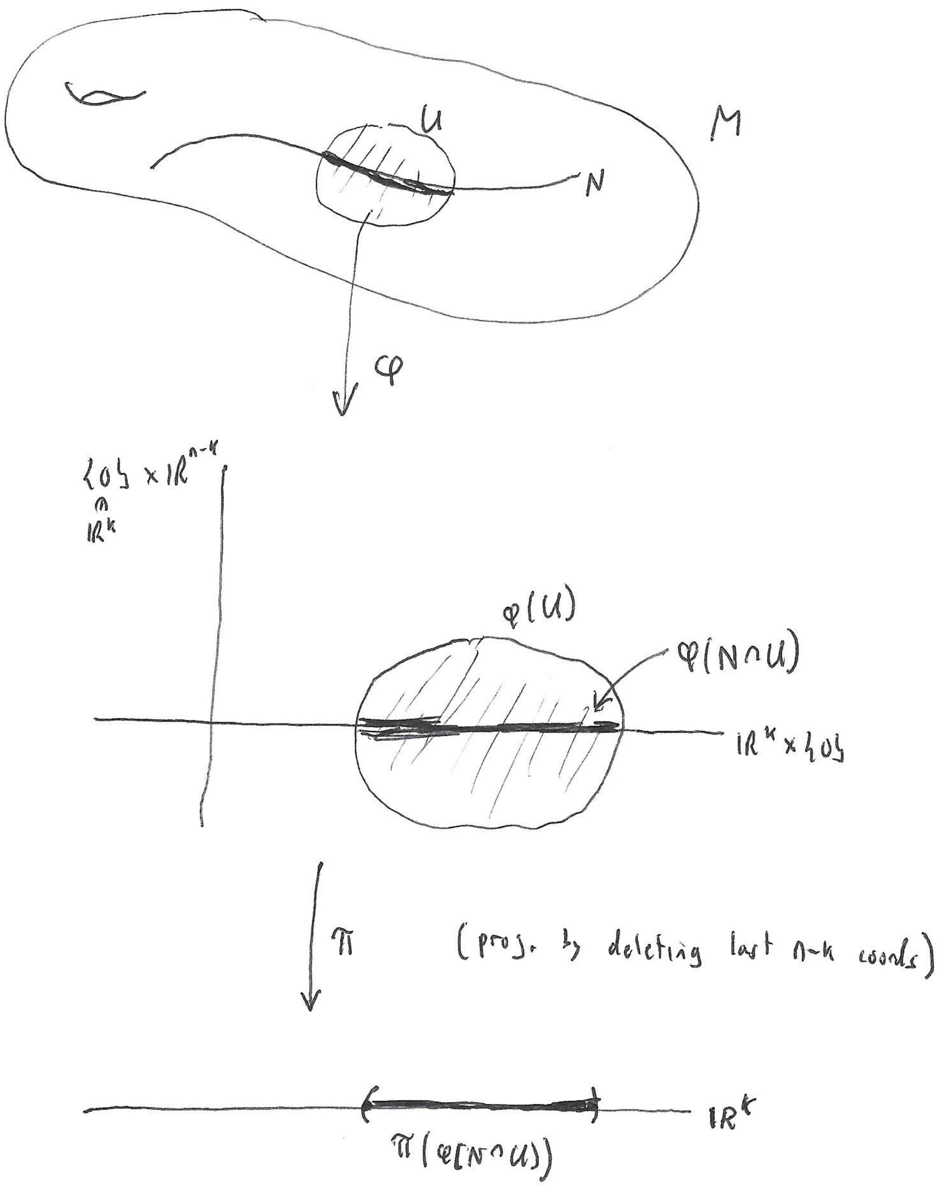
**General idea**: A submfd of a mfd $M$ is a subset $N$ that is itself a mfd, but with topology and smooth structure induced from $M$.

**Def:** Let $M$ be a (smooth) mfd., $n = \dim(M)$. Let $N \subset M$, $N \neq \emptyset$.

Call a chart $(U, \varphi)$ of $M$ *adapted to $N$* if, for some $k \leq n$,

$$\varphi(N \cap U) = \{(x^1, \dots, x^n) \in \varphi(U) \mid x^{k+1} = \dots = x^n = 0\}$$

$$= \varphi(U) \cap \underbrace{(\mathbb{R}^k \times \{0\})}_{\text{(interp. as } \mathbb{R}^n \text{ if } k=n)}, \quad \{0\} \subset \mathbb{R}^{n-k}$$



<u>Def</u>: Let $M, N$ be as above. The set $N$ is a <u>submanifold</u> of $M$ if, $\forall p \in N$, $\exists$ chart $(U, \varphi)$ of $M$, adapted to $N$, with $p \in U$.

<u>Remark</u>: This is one reason that <u>maximal</u> atlases are important. An arbitrary atlas of $M$, e.g. one with finitely many charts, may not have charts that are adapted to a desired submfd $N$.

Now suppose $(U, \varphi)$ chart of $M$ adapted to $N$, with "$k$" as earlier. Define a $k$-dim'l chart of $N$ as follows. Let

$V = N \cap U$, an open set in relative topology on $N$ (the topology on $N$ induced by topology of $M$)

$\psi = \pi \circ \varphi|_V : V \to \mathbb{R}^k$

($\pi$ is projection $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$; see earlier diagram)

Since $\varphi(U)$ open in $\mathbb{R}^n$, have

$$\underbrace{\varphi(U) \cap (\mathbb{R}^k \times \{0\})}_{= \varphi(V)} \text{ open in } \mathbb{R}^k \times \{0\},$$

$$\Rightarrow \psi(V) = \pi(\varphi(V)) \text{ open in } \mathbb{R}^k.$$

~~(in fact, $\psi : V \to \psi(V)$ is a homeomorphism)~~.

Clearly $\psi : V \to \psi(V)$ is bijective (in fact a homeo).

$\therefore (V, \psi)$ is a $k$-dim'l chart of $N$.

What is $\underbrace{\psi^{-1}}$?

Here, regard $\psi$ as map $V \to \psi(V)$, a bijection..

<u>Claim</u>: $\psi^{-1} = \varphi^{-1} \circ i$,

where $i: \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is the map $x \longmapsto (x, \underset{\mathbb{R}^{n-k}}{\underset{\cap}{0}})$.

<u>Pf</u>: For $p \in V$,

$$\varphi^{-1} \circ i \circ \psi(p) = \varphi^{-1} \circ \underbrace{i \circ \pi}_{= \text{id. on } \mathbb{R}^k \times \{0\}} \circ \underbrace{\varphi(p)}_{\in \mathbb{R}^k \times \{0\}}$$

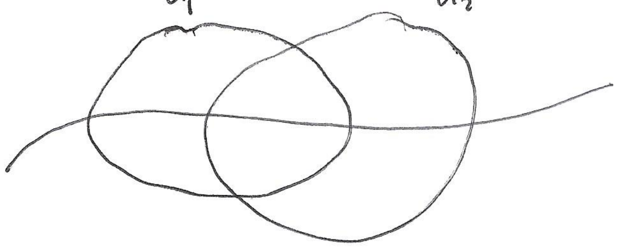
$$= \varphi^{-1}(\varphi(p)) = p.$$

Thus $(\varphi^{-1} \circ i) \circ \psi = id_V$.

Since $\psi$ bijective (as map $V \to \psi(V)$), this implies $\psi^{-1} = \varphi^{-1} \circ i$.

$\boxtimes$

Now suppose $N$ is a submfd of $M$; then $N$ covered by domains of adapted charts. For $i = 1,2$, let $(U_i, \varphi_i)$ be charts of $M$ adapted to $N$, and let $(V_i, \psi_i)$ be the corresp. $k_i$-dim'l charts of $N$




Note that $V_1 \cap V_2$ is open in $V_1$. Hence $\psi_1(V_1 \cap V_2)$ open in $\psi(V_1)$, which is open in $\mathbb{R}^{k_1}$, so $\psi_1(V_1 \cap V_2)$ open in $\mathbb{R}^{k_1}$.

Consider overlap-map $\psi_2 \circ \psi_1^{-1} : \underbrace{\psi_1(V_1 \cap V_2)}_{\subset \mathbb{R}^{k_1}} \longrightarrow \underbrace{\psi_2(V_1 \cap V_2)}_{\subset \mathbb{R}^{k_2}}$

Put subscripts 1,2 on maps $\pi$, $i$ accordingly. Have

$$\psi_2 \circ \psi_1^{-1}\Big|_{\psi_1(V_1 \cap V_2)} = \underbrace{\pi_2}_{\substack{C^\infty \\ \text{from} \\ \mathbb{R}^n \\ \text{to} \\ \mathbb{R}^{k_2}}} \circ \underbrace{\varphi_2 \circ \varphi_1^{-1}}_{\substack{C^\infty \\ \text{from open} \\ \text{set in} \\ \mathbb{R}^n \text{ to open set in } \mathbb{R}^n}} \circ \underbrace{\left( i_1 \Big|_{\psi_1(V_1 \cap V_2)} \right)}_{C^\infty \text{ from open set in } \mathbb{R}^{k_1} \text{ to } \mathbb{R}^n.}$$

By Chain Rule Thm, $\psi_2 \circ \psi_1^{-1}\Big|_{\psi_1(V_1 \cap V_2)}$ is $C^\infty$ : $(\text{open} \subset \mathbb{R}^{k_1}) \to (\text{open} \subset \mathbb{R}^{k_2})$.

Similarly, its inverse $\psi_1 \circ \psi_2^{-1}\Big|_{\psi_2(V_1 \cap V_2)}$ is $C^\infty$ : $(\text{open} \subset \mathbb{R}^{k_2}) \to (\text{open} \subset \mathbb{R}^{k_1})$.

Hence $k_1 = k_2$, and the overlap-maps are smooth.

$\therefore$ Collection of charts $\{(V, \psi)\}$ obtained from $N$-adapted charts of $M$, as above, is an atlas on $M$, and the topology on $N$ induced by this atlas is the relative topology on $N$ (as a subset of $M$).

~~In partic., N is a manifold (with the above atlas).~~

In particular, a submfd of $M$ is a <u>manifold</u>, with topology and smooth structure induced from $M$.

If $N$ not connected, then different connected components could have different dimensions. But usually we're interested only in case where all components have the same dimension. In that case,

$$\text{codimension of } N \text{ in } M \underset{\text{def}}{=} \dim(M) - \dim(N).$$