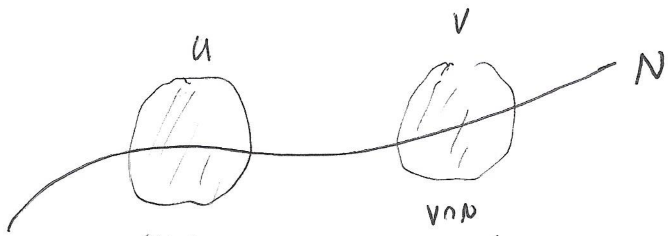
13.1 7/27/17

Remark: "N is a submfd of M" is a $\overbrace{\text{local}}^{\text{local in } M}$ condition on N:

Prop: Let M be a mfd, $N \subset M$ nonempty. Suppose that $\forall p \in N$, $\exists$ open set $U \subset M$, containing p, s.t. $N \cap U$ is a submfd. of U. Then N is a submfd of M.

Pf: HW. $\boxtimes$

Picture:



$U \cap N$ submfd of U

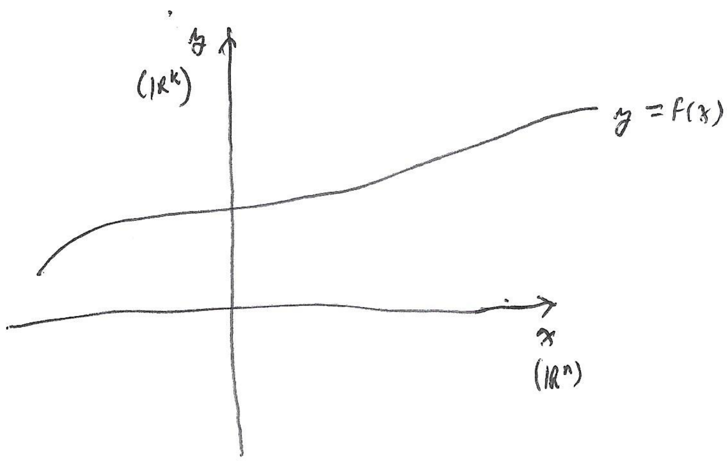
$V \cap N$ submfd of V.

Submanifolds of $\mathbb{R}^n$

Recall: If $f: X \to Y$ is any function, the graph of f is the set $\{(x, f(x)) \mid x \in X\} \subset X \times Y$

Example/Prop.: Let $f: (U \overset{\text{open}}{\subset} \mathbb{R}^n) \to \mathbb{R}^k$ be smooth. Then the graph of f is a submanifold of $\mathbb{R}^n \times \mathbb{R}^k$ ($\leftrightarrow \mathbb{R}^{n+k}$), of dimension n and codimension k.

Pf:



Let N = graph of f. Define $\varphi: U \times \mathbb{R}^k \to U \times \mathbb{R}^k$ by

$$\varphi(x, y) = (x, y - f(x))$$

Obviously, φ is 1-1; easily seen to be smooth as well. Jacobian is

$$J_\varphi(x, y) = \begin{pmatrix} I_{n\times n} & O_{n\times k} \\ \underset{-J_f(x)}{*} & I_{k\times k} \end{pmatrix}$$, invertible since $\det = 1$.

So, by Inverse Fcn Thm, φ is a local diffeo. But φ also 1-1, so φ is a (global) diffeo $U \times \mathbb{R}^k \longrightarrow U \times \mathbb{R}^k \subset \mathbb{R}^{n+k}$.

Thus $(U \times \mathbb{R}^k, \varphi)$ is a chart of $\mathbb{R}^{n+k}$ (with its standard smooth structure).

Have $N \subset U \times \mathbb{R}^k$, so $N = N \cap (U \times \mathbb{R}^k)$,

and

$$\begin{aligned} \varphi(N \cap (U \times \mathbb{R}^k)) = \varphi(N) &= \{(x, y - f(x)) \mid x \in U,\ y = f(x)\} \\ &= \{(x, 0) \mid x \in U\} \\ &= U \times \{0\}. \end{aligned}$$

But also

$$\varphi(U \times \mathbb{R}^k) \cap (\mathbb{R}^n \times \{0\}) = (U \times \mathbb{R}^k) \cap (\mathbb{R}^n \times \{0\}) = U \times \{0\}.$$

Hence

$$\varphi(N \cap (U \times \mathbb{R}^k)) = \varphi(N) \cap (\mathbb{R}^n \times \{0\}).$$

Thus $(U \times \mathbb{R}^k, \varphi)$ is adapted to N, and $U \times \mathbb{R}^k$ covers N.

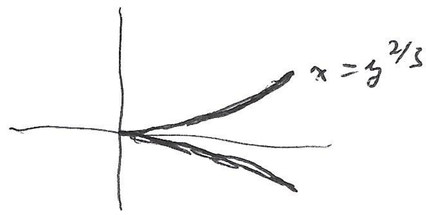
Thus N is a submfd of $U \times \mathbb{R}^k$, hence of $\mathbb{R}^{n+k}$.

(The mfds N, $U \times \mathbb{R}^k$, $\mathbb{R}^{n+k}$ all have 1-chart atlases we're using.)

Statements about dim. and codim. are immediate.

$\boxtimes$

<u>Ex</u>: Define $f: \mathbb{R} \to \mathbb{R}^2$ by $f(t) = (t^2, t^3)$. Then f is smooth. The <u>image</u> of f is



, not a "smooth object". (Exercise: Show that the image is not a submfd of $\mathbb{R}^2$.) But, from Prop. above, the <u>graph</u> of f is a submfd of $\mathbb{R}^3$.

<u>Def</u>: Let $F: (U^{\text{open}} \subset \mathbb{R}^n) \to \mathbb{R}^m$ be smooth. A <u>critical point</u> of F is a pt. $p \in U$ s.t. $DF|_p : \mathbb{R}^n \to \mathbb{R}^m$ is not surjective. A <u>critical value</u> of F is a pt. $c \in \mathbb{R}^m$ s.t. $F^{-1}(\{c\})$ contains a critical point. A <u>regular value</u> of F is any $q \in \mathbb{R}^m$ that is not a crit. value. (In partic., every $q \notin \text{image}(F)$ is a reg. value.)

<u>Ex</u>: Define $F: \mathbb{R}^2 \to \mathbb{R}$ by
$$F(x,y) = x^2 - y^2 + 1.$$
Then $J_F(x,y) = (2x, 2y)$. Associated lin. map $\mathbb{R}^2 \to \mathbb{R}$ is surjective unless $x = y = 0$. Thus the only crit pt. is $(0,0)$, and the only crit. value is 1.

<u>Note</u>: With notation as in Def., if $n < m$ then

(i) Every $p \in U$ is a crit. pt.

(ii) " $q \in \mathrm{Im}(F)$ is a crit. value.

(iii) " $q \notin \mathrm{Im}(F)$ " " reg. value.
$\quad\;\; \cap$
$\quad\;\; \mathbb{R}^m$

Just FYI { **Def:** A <u>hypersurface</u> is a codim-1 submfd.

Prop: Let $f : (U \overset{\text{open}}{\subset} \mathbb{R}^n) \to \mathbb{R}$ be smooth. If c is a reg. value of f, then $f^{-1}(c)$ is either empty or a codim-1 submfd of $\mathbb{R}^n$.

Abuse of notation for $f^{-1}(\{c\})$

Pf: By replacing f with $f-c$, may assume $c=0$.

If $f^{-1}(0)$ empty, nothing to prove, so assume non-empty.

Let $p \in f^{-1}(0)$. A linear map $T: \mathbb{R}^n \to \mathbb{R}$ is surjective unless $T=0$. Since $Df|_p$ is surjective (by hyp. that c is a reg. value), $Df|_p \neq 0$, $\Rightarrow J_f(p) \neq 0$, so $\exists\, i$ s.t. $\frac{\partial f}{\partial x^i}(p) \neq 0$.

~~Idea will be to replace last coord fcn~~

For concreteness (and WLOG), assume $\frac{\partial f}{\partial x^n}(p) \neq 0$.

Idea will be to replace coord-fcn x^n by f (which is 0 on $f^{-1}(0)$) and get a chart adapted to $f^{-1}(0)$.

For $x \in \mathbb{R}^n$, write $x = (\vec{x}, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Define $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\varphi(\vec{x}, y) = (\vec{x}, f(\vec{x}, y))$$

Then

$$J_\varphi(\vec{x}, y) = \begin{array}{cc} \begin{array}{cc} \overset{n-1}{\longleftrightarrow} & \overset{1}{\longleftrightarrow} \end{array} & \\ \left(\begin{array}{ccc|c} & I_{(n-1)\times(n-1)} & & 0 \\ \hline \frac{\partial f}{\partial x^1} & \cdots & \frac{\partial f}{\partial x^{n-1}} & \frac{\partial f}{\partial x^n} \end{array}\right) & \begin{array}{l} \updownarrow n-1 \\ \updownarrow 1 \end{array} \end{array}$$

Thus $\det(J_\varphi(p)) = \frac{\partial f}{\partial x^n}(p) \neq 0$,

$\Rightarrow J_\varphi(p)$ invertible.

By Inverse Fcn Thm, φ is a local diffeo.

Let U be an open nbhd of p, small enough that $\varphi|_U$ is a diffeo. Write just "φ" for "$\varphi|_U$" (remembering that "N submfd of M" is a local condition). Then

$$f^{-1}(0) \cap U = \{x \in U \mid f(x) = 0\}$$
$$= \{x \in U \mid (\text{last coord. of } \varphi(x)) = 0\},$$

$\Rightarrow$

$$\varphi(f^{-1}(0) \cap U) = \{\varphi(x) \in \varphi(U) \mid (\text{last coord of } \varphi(x)) = 0\}$$
$$= \{q \in \varphi(U) \mid x^n(q) = 0\}$$
$$= \varphi(U) \cap (\mathbb{R}^{n-1} \times \{0\}).$$

Hence (U, φ) is adapted to $f^{-1}(0)$, and $p \in U$.

Since p was arb., this proves that $f^{-1}(0)$ is a submfd of $\mathbb{R}^n$, of codimension 1.

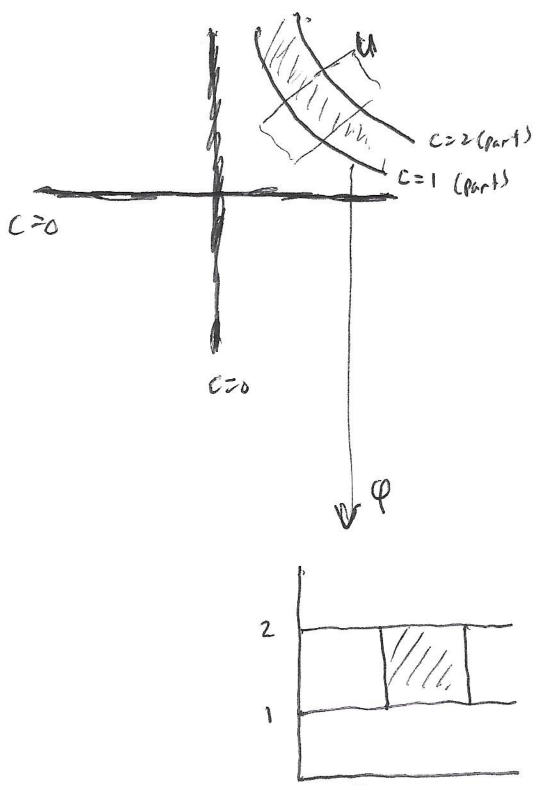
$\blacksquare$

Ex./Cor: (a) S^n is an n-dim'l submfd of $\mathbb{R}^{n+1}$

(b) For $c \neq 0$, the set $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = c\}$ is a 2-dim. submfd of $\mathbb{R}^3$. (The only crit. pt of $f : (x,y,z) \mapsto x^2+y^2-z^2$ is $(0,0,0)$, which is $\notin f^{-1}(c)$.) Note that the number of connected components of $f^{-1}(c)$ depends on c in this example.

Ex: $n=2$, $f(x,y) = xy$

Only crit. value is 0. (And only crit. pt. is $(0,0)$).



Via map φ in proof, We're re-coordinatizing $\mathbb{R}^2$ s.t. the direction "transverse" to hyperbolic arcs is 2nd coord:

$$\varphi(x,y) = (x, xy)$$

$$\text{quadrant I} \xrightarrow[1\text{-}1]{} \text{quadrant I}.$$

Ex: $n=2$, $f(x,y) = x^2+y^2$. Same crit pt. & crit. val. as in previous example.

Take $c = a^2$, $a>0$. Picture below illustrates the map φ for the indicated set U