The proposition from last lecture (p. 13.5) generalizes to the following:

<u>Regular Value Thm, version 1</u>: Let $f : (U^{open} \subset \mathbb{R}^n) \to \mathbb{R}^m$ be smooth. Then the inverse image of any regular value is either empty, or a submfd of $\mathbb{R}^n$ of codimension m.

<u>Pf</u> (Sketch) Let $c \in \mathbb{R}^m$ be a reg. value of f. By replacing f with $f - c$, may assume $c = \underset{\mathbb{R}^m}{\underset{\cap}{0}}$. If $f^{-1}(0)$ empty, have nothing to prove, so assume nonempty.

Let $p \in f^{-1}(0)$. Then

$$Df|_p \text{ surjective} \Rightarrow \operatorname{rank}(J_f(p)) = m$$
$$\Rightarrow \text{some set of } m \text{ columns of } J_f(p) \text{ spans } \mathbb{R}^m.$$

~~WLOG, assume~~ (Note that since p is a reg. value, we are automatically assuming $n \geq m$.) WLOG, assume the last m columns of $J_f(p)$ span $\mathbb{R}^m$. Then

$$J_f(p) = \left(\underset{\longleftrightarrow \; n-m}{*} \;\middle|\; \underbrace{\underset{\longleftrightarrow \; m}{*}}_{\text{last } m \text{ columns form an invertible } m \times m \text{ matrix.}} \right) \updownarrow m$$

Write $\mathbb{R}^n$ as $\mathbb{R}^{n-m} \times \mathbb{R}^m$ (omit 1st factor if $n = m$).

" $x \in \mathbb{R}^n$ " $(\vec{x}, \vec{y})$.

Define $\varphi : U \to \mathbb{R}^n$ by

$$\varphi(\vec{x}, \vec{y}) = (\vec{x}, f(\vec{x}, \vec{y})).$$

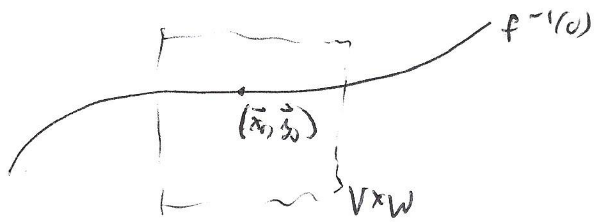
The rest of pf. is left as an exercise. (Proceed analogously to pf of $m=1$ case given in lecture 13.)

<u>Alternative pf</u>: Proceed as in 1st pf, up to bottom of previous page.

The invertibility of the right-hand $m \times m$ block in $J_f(p)$ implies that $(D^{[2]}f)\big|_p : \mathbb{R}^m \to \mathbb{R}^m$ is an isomorphism (where notation "$D^{[2]}f$" is as in Implicit Fcn Thm). Write p as $(\underset{\underset{\mathbb{R}^{n-m}}{\cap}}{\vec{x}_0}, \underset{\underset{\mathbb{R}^{m}}{\cap}}{\vec{y}_0})$.

By Implicit Fcn Thm, $\exists$ open nbhds $\underset{\underset{\mathbb{R}^{n-m}}{\cap}}{V}$ of $\vec{x}_0$, $\underset{\underset{\mathbb{R}^{m}}{\cap}}{W}$ of $\vec{y}_0$, and a smooth map $g : V \to W$, s.t. for all $(\vec{x}, \vec{y}) \in V \times W$,

$$f(\vec{x}, \vec{y}) = 0 \iff \vec{y} = g(\vec{x})$$



Thus $f^{-1}(0) \cap (V \times W) =$ graph of g. Since g smooth, its graph is a submfd of $\underset{\underset{\mathbb{R}^{n}}{\cap}}{\underbrace{V \times W}}$ of dimension $n-m$, hence of codimension m.

(See lecture 13 notes.) Since p was arb. elt. of $f^{-1}(0)$, it follows that $f^{-1}(0)$ is a submfd of $\mathbb{R}^n$ of codim. m. (See p. 13.1.) $\blacksquare$

<u>Remark</u>: The Implicit Fcn Thm is really what's at the heart of the Reg. Value Thm. The level-set $f^{-1}(c)$ is the sol'n-set of

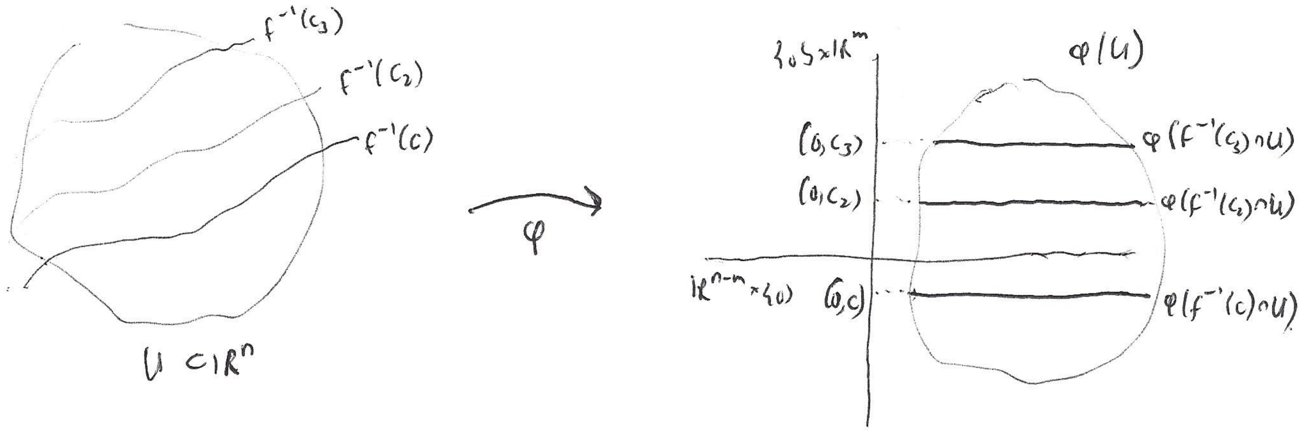
$$\left.\begin{aligned} f^1(\vec{x},\vec{y}) &= c^1 \\ f^2(\vec{x},\vec{y}) &= c^2 \\ &\vdots \\ f^m(\vec{x},\vec{y}) &= c^m, \end{aligned}\right\} \quad (*)$$

where $f = \begin{pmatrix} f^1 \\ \vdots \\ f^m \end{pmatrix}$, $c = \begin{pmatrix} c^1 \\ \vdots \\ c^m \end{pmatrix}$. The Implicit Fcn. Thm essentially says that each of the m scalar eqs. in $(*)$ knocks out one degree of freedom from a small enough n-dim'l open nbhd of p (assuming p is not a crit. pt.), leaving a "smooth $(n-m)$-dim'l set" as the sol'n set.

The first proof, using the Inverse Function Thm, is still worthwhile; it leads to a nice picture (given shortly). First note that the condition "p is not a crit pt. of f" is an <u>open</u> condition on p: if the $m\times m$ submatrix of $J_f(p)$ used in pf is invertible, then the $m\times m$ submatrix of $J_f(q)$ formed by the same columns is invertible for all q suff. close to p. Hence, if c is a reg. value of f, and $p\in f^{-1}(c)$, and U is a nbhd of p s.t. the map $\varphi: U\to\mathbb{R}^n$ in 1st proof is a diffeo. onto its image, then $f^{-1}(c')\cap U$ is a codim-m submfd of U for all c' suff. close to c.

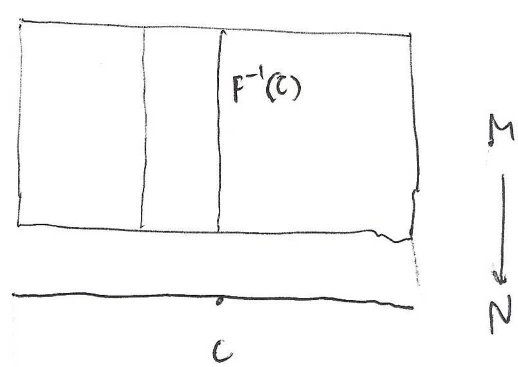
(cont'd next page)

Thus, a picture of $\varphi|_U$ is



So φ "straightens out" the coord. system on $\mathbb{R}^n$ (near p), in such a way that level-sets of f correspond to level-sets of the projection $\mathbb{R}^{n-m}\times\mathbb{R}^m \longrightarrow \mathbb{R}^m$.

In the picture above, the (images of) these level sets of f were depicted horizontally, to match what was given by our ordering of variables in $\mathbb{R}^n$. Customarily, given a map of mfds $F: M \to N$, level-sets of F are drawn vertically instead of horizontally:



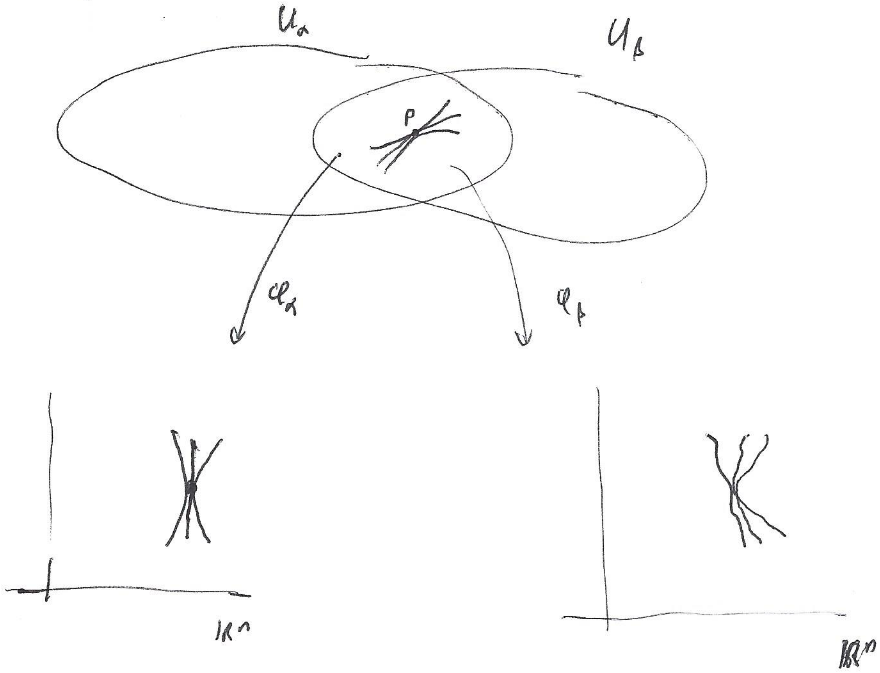
The sets $F^{-1}(c)$ are called <u>fibers</u> of the map f.

Tangent spaces

Def: Let M be a manifold. A (smooth) (parametrized) curve based at $p \in M$ is a smooth map $\gamma : I \to M$, where I is an open interval containing 0, and $\gamma(0) = p$.

As noted before, the image of γ can be a "non-smooth object", or even just a point. (But usually we depict curves as "geometrically smooth" images.)

Fix $p \in M$. We have an intuitive notion of tangency of curves based at $p \in M$.



Idea is that γ_1, γ_2 based at $p \in M$ should be called tangent if $\varphi \circ \gamma_1$, $\varphi \circ \gamma_2$ are tangent to each other at $\varphi(p) \in \mathbb{R}^n$ for some, maybe any, chart (U, φ) containing $\mathrm{img}(\gamma_1), \mathrm{img}(\gamma_2)$. But remember that $\varphi \circ \gamma_i$ need not be a geometrically smooth object, so our intuitive notion of tangency needs some adjustment.

First, make notion of tangency precise for curves in $\mathbb{R}^n$.

Def: Two curves γ_1, γ_2 based at $p \in \mathbb{R}^n$ are tangent at p if $\gamma_1'(0) = \gamma_2'(0)$, where $\gamma'(t) = \frac{d}{dt}\gamma(t) \in \mathbb{R}^n$.

$$= D\gamma|_t (1).$$

Note difference between this and the "geometric" notion of tangency, which would require only $\gamma_1'(0) \propto \gamma_2'(0)$, but would require both ~~~~ $\gamma_1'(0), \gamma_2'(0)$ to be non zero.

Write $\gamma_1 \sim \gamma_2$ if γ_1, γ_2 tangent at p.
Clearly "$\sim$" is an equiv. rel'n. Denote equiv. class of γ by $[\gamma]$.

Def: A tangent vector at $p \in \mathbb{R}^n$ is an equivalence class $[\gamma]$ under the relation above. The tangent space of $\mathbb{R}^n$ at p, as a set, is the set of all tangent vectors at p.

To transfer these notions to manifold M, let $p \in M$, and let $(U_\alpha, \varphi_\alpha)$ be a chart with $p \in U_\alpha$. Let γ_1, γ_2 be two curves based at p. Note that "$(\varphi_\alpha \circ \gamma_i)'(0)$" makes sense whether or not $\operatorname{img}(\gamma_i) \subset U_\alpha$, since $\gamma_i(t) \in U_\alpha$ for $|t|$ suff. small. Write

$$\gamma_1 \sim_\alpha \gamma_2 \quad \text{iff} \quad (\varphi_\alpha \circ \gamma_1)'(0) = (\varphi_\alpha \circ \gamma_2)'(0);$$

obviously $\sim_\alpha$ is an equiv. rel'n. If (U_β, φ_β) is another chart with $p \in U_\beta$, and $\gamma_1 \sim_\alpha \gamma_2$, do we have $\gamma_1 \sim_\beta \gamma_2$?

Yes:

$$(\varphi_\beta \circ \gamma_1)'(0) = (\varphi_\beta \circ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ \gamma_1)'(0)$$

$$= \underbrace{J_{\varphi_\beta \circ \varphi_\alpha^{-1}}(\varphi_\alpha(p))}_{\substack{n\times n \text{ matrix,} \\ \text{where} \\ n = \dim(M)}} \underbrace{(\varphi_\alpha \circ \gamma_1)'(0)}_{\substack{\text{column vector} \\ \text{in } \mathbb{R}^n}}$$

$$= J_{\varphi_\beta \circ \varphi_\alpha^{-1}}(\varphi_\alpha(p))(\varphi_\alpha \circ \gamma_2)'(0)$$

$$= (\varphi_\beta \circ \gamma_2)'(0) \quad \text{by reversing above steps.}$$

Thus $\gamma_1 \sim_\alpha \gamma_2 \iff \gamma_1 \sim_\beta \gamma_2$. So we may simply write "$\gamma_1 \sim \gamma_2$" if $\gamma_1 \sim_\alpha \gamma_2$ for <u>some</u> chart $(U_\alpha, \varphi_\alpha)$ containing p; the equiv. rel'n is indep. of choice of chart.

<u>Def</u>: Let γ_1, γ_2 be curves based at $p \in M$. Define $\gamma_1 \sim \gamma_2$ if $\gamma_1 \sim_\alpha \gamma_2$ for some (hence any) chart $(U_\alpha, \varphi_\alpha)$ with $p \in U_\alpha$. (As seen above, $\sim$ is an equiv. rel'n on the set of curves based at p.) Write equiv. class of γ as $[\gamma]$.

<u>Def</u>: The <u>tangent space of M at p</u>, as a set, is

$$T_pM = \{\text{curves based at } p\}/\sim$$

Vector-space structure on T_pM

Fix $q \in \mathbb{R}^n$. For each $v \in \mathbb{R}^n$, define a curve γ_v based at q by

$$\gamma_v(t) = q + tv, \quad t \in \mathbb{R}.$$

(The notation γ_v is temporary.) Observe that γ_v is simply a parametrized straight line through q, and that $\gamma_v'(0) = v$.

Since $\gamma_v'(0) = v$, the map

$$\begin{aligned} \mathbb{R}^n &\longrightarrow T_q\mathbb{R}^n \\ v &\longmapsto [\gamma_v] \end{aligned}$$

is a bijection. We use this bijection to endow $T_q\mathbb{R}^n$ with a vector-space structure:

$$[\gamma_v] + [\gamma_w] \underset{\text{def}}{=} [\gamma_{v+w}], \quad v, w \in \mathbb{R}^n$$

$$c[\gamma_v] \underset{\text{def}}{=} [\gamma_{cv}], \quad c \in \mathbb{R}, v \in \mathbb{R}^n.$$

Observe that $[\gamma_{\underset{\substack{\text{zero-elt.} \\ \text{of } \mathbb{R}^n}}{0}}]$ is the zero-elt. of $T_q\mathbb{R}^n$.

(I.e. the <u>constant curve</u> is a representative of the zero-elt. of $T_q\mathbb{R}^n$.)

We now attempt to transfer this vector-space structure to T_pM (M a manifold).

(To be continued next lecture.)