Differential Geometry—MTG 6256
Point-Set Topology: Glossary and Review

Organization of this glossary. Several general areas are covered: sets and functions, topological spaces, metric spaces (very special topological spaces), and normed vector spaces (very special metric spaces). In order to proceed from concrete objects to more general objects, topological spaces appear last. Because of this, several definitions appear both for metric spaces and for topological spaces even though the second definition implies the first. However, to limit the length of these notes, certain properties relevant to metric spaces have had their definitions deferred to the topological-space section.

In the section on normed vector spaces, material on the operator norm is included, although strictly speaking it's not part of point-set topology.

1 Set and Function Terminology

image; inverse image. Let A and B be sets and $f : A \to B$ a function. If $U \subset A$, then the *image of* U *under* f, denoted $f(U)$, is defined by $f(U) = \{b \in B \mid b = f(u) \text{ for some } u \in U\}$. If $V \subset B$, then the *inverse image* (or *pre-image*) *of* V *under* f, denoted $f^{-1}(V)$, is defined by $f^{-1}(V) = \{a \in A \mid f(a) \in V\}$. The inverse image of a set is *always* defined, regardless of whether an inverse function exists.

One always has $f(f^{-1}(V)) = V$ and $f^{-1}(f(U)) \supset U$, but in general $f^{-1}(f(U)) \neq U$. One also has $f^{-1}(U \bigcup V) = f^{-1}(U) \bigcup f^{-1}(V)$, $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$, and $f(U \bigcup V) = f(U) \bigcup f(V)$. However, while $f(U \cap V) \subset f(U) \cap f(V)$, in general $f(U \cap V) \neq f(U) \cap f(V)$.

complement. If A is a set and $B \subset A$, the *complement* of B in A, denoted $A - B$ or B' in these notes, consists of all elements of A not in B.

disjoint. Two sets U, V are *disjoint* if they have no elements in common ($U \cap V = \emptyset$).

disjoint union. The *disjoint union* $U \coprod V$ is the union of two disjoint sets U, V; the notation $U \coprod V$ simply means $U \cup V$ together with an assertion (or reminder) that U and V are disjoint. The disjoint union of any collection of (disjoint) sets is defined and denoted similarly.

equivalence relation. An *equivalence relation* on a set A is a relation $\sim$ that is *reflexive* ($a \sim a \ \forall a \in A$), *symmetric* ($a \sim b$ implies $b \sim a$), and *transitive* ($a \sim b$ and $b \sim c$ implies $a \sim c$). The *equivalence class* of an element $a \in A$ is the set of all elements b such that $a \sim b$. An equivalence relation on A partitions A into a disjoint union of equivalence classes.

power set. Let X be a set. The *power set of* X, which we will denote $\mathcal{P}(X)$, is the set of all subsets of X.

1.1 Metric Space Notions

metric. A *metric* on a (necessarily nonempty) set X is a function $d : X \times X \to \mathbf{R}$ satisfying the following three conditions:

(i) *symmetry*: $d(x,y) = d(y,x) \ \ \forall x,y \in X$

(ii) *triangle inequality*: $d(x,z) \leq d(x,y) + d(y,z) \ \ \forall x,y,z \in X$.

(iii) *positivity*: $d(x,y) \geq 0 \ \ \forall x,y, \in X$, and $d(x,y) = 0$ iff $x = y$.

metric space. A *metric space* is a pair (X,d), where X is a nonempty set and d is a metric on X. *Examples:* (i) $X = \mathbf{R}^n$ with $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$, where $\|\mathbf{x}\| = (\sum_i (x^i)^2)^{1/2}$ and where $\mathbf{x} = (x^1, \ldots, x^n)$.[1] This is called the *standard metric* or *Euclidean metric* on $\mathbf{R}^n$. In these notes, whenever $\mathbf{R}^n$ or a subset of $\mathbf{R}^n$ is referred to as a metric space, the standard metric is intended unless otherwise specified. (ii) Any nonempty subset of $\mathbf{R}^n$ with d inherited from $\mathbf{R}^n$ (e.g. the unit sphere $\{\mathbf{x} \in \mathbf{R}^3 \mid \|\mathbf{x}\| = 1\}$, with $d(\mathbf{x},\mathbf{y})$ equal to the length of the line segment in $\mathbf{R}^3$ joining $\mathbf{x}$ and $\mathbf{y}$). (iii) Any nonempty subset Y of a metric space (X,d), with d inherited from X. In this case d (restricted to Y) is called the *induced metric* on Y.

Note that a metric space need not have any linear structure; in general there is no such thing as the sum of two points in a metric space. Metric spaces are more general than the normed vector spaces considered below.

Common notational simplification. In statements for which the function d does not need to be referenced explicitly, one often refers simply to X as a metric space, rather than writing (X,d). This convention will be used below.

convergence; limit of a sequence. A sequence $\{x_n\}$ of points in a metric space (X,d) *converges* if there exists $y \in X$ for which $\lim_{n\to\infty} d(x_n,y) = 0$. In this case we say that $\{x_n\}$ *converges to* y (also written $x_n \to y$) and that y is the *limit* of the sequence $\{x_n\}$. *Examples.* (i) $X = \mathbf{R}, d(x,y) = |x-y|, x_n = 1/n$. This sequence converges to 0. (ii) Same as (i), except with $X = \mathbf{R} - \{0\}$ (the "punctured" line). In this case $\{x_n\}$ does not converge. (We *don't* say that $\{x_n\}$ converges to a point not in X.)

continuous function. If X is a metric space, $f : X \to \mathbf{R}$ is a function, and $x \in X$, we say f is *continuous* at x if for every sequence $\{x_n\}$ in X with limit y, we have $\lim_{n\to\infty} f(x_n) = f(x)$. If f is continuous at each point, we simply say that f is continuous.

More generally, the same definition of continuity applies to functions from X to any other metric space.

Warning: The definition of continuity above is special to *metric spaces.*

[1] For reasons to be discussed later in this course, we will denote coordinates in $\mathbf{R}^n$ using superscripts rather than subscripts. For typographical simplicity, elements of $\mathbf{R}^n$ will usually be written as ordered n-tuples (regarded as row vectors for linear-algebraic purposes), but for purposes of matrix operations they should be written as column vectors.

Cauchy sequence. A *Cauchy sequence* in a metric space (X, d) is a sequence of points $\{x_n\}$ in X with the following property: $\forall \epsilon > 0$, $\exists N$ such that $\forall n, m \geq N$, $d(x_n, x_m) < \epsilon$.

Fact. "Cauchy" is a necessary condition for a sequence in a metric space to converge. However, example (ii) under "convergence; limit of a sequence" shows that this condition is *not sufficient* to ensure convergence.

dense subset. A subset U of a metric space (X, d) is *dense* (in X) if for all $x \in X$ there exists a sequence $\{u_n\}$ in U with $u_n \to x$.

complete. A metric space (X, d) is *complete* if every Cauchy sequence converges. A subset $Y \subset X$ is called complete if the metric space (Y, d) is complete (in other words, if Y is complete with respect to the induced metric). *Examples.* (i) $\mathbf{R}^n$ is complete. (ii) $\mathbf{R}^n - \{0\}$ is not complete. (iii) The disk $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$ is complete. (iv) The disk $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$ (strict inequality this time) is not complete.

open and closed balls; disks. Let (X, d) be a metric space, $p \in X$, $r > 0$. The *open ball* of radius r centered at p is the set $B_r(p) := \{x \in X \mid d(x, p) < r\}$.
The *closed ball* of radius r centered at p is the set $\overline{B}_r(p) := \{x \in X \mid d(x, p) \leq r\}$. *Disk* is a synonym for ball.

open and closed sets. A subset U of a metric space is called *open* if $\forall p \in U$ $\exists r > 0$ such that $B_r(p) \subset U$. A subset U is called *closed* if its complement is open. *Examples:* (1) Open balls are open sets. (2) Closed balls are closed sets. (3) The set obtained by deleting one point from the boundary of closed ball in $\mathbf{R}^n$ $(n > 0)$ is neither open nor closed.

In general, most subsets are neither open nor closed.

relation of "closed" to "complete". *Facts.* (1) Every complete subset of a metric space is closed. (2) Every closed subset of a *complete* metric space is complete.

Thus, in a complete metric space, closed subsets and complete subsets are the same.

bounded set. A subset U of a metric space is *bounded* if U is contained in a ball $B_r(p)$ for some r, p.

equivalent metrics. Two metrics d, d' on the same set X are *equivalent* if there exist $c_1 > 0, c_2 > 0$ such that $c_1 d(x, y) \leq d'(x, y) \leq c_2 d(x, y)$ $\forall x, y \in X$.

Fact. Equivalent metrics determine the same open sets and the same Cauchy sequences. (Easy to show.)

1.2 Normed Vector Spaces

Below, all vector spaces are assumed real, but everything generalizes to the complex case.

norm. A *norm* $\| \ \|$ on a vector space V is a real-valued function on V satisfying the following three conditions:

(i) *homogeneity*: $\|cv\| = |c|\|v\| \ \ \forall v \in V, c \in \mathbf{R}$.

(ii) *triangle inequality*: $\|v + w\| \leq \|v\| + \|w\| \ \ \forall v, w \in V$.

(iii) *positivity*: $\|v\| \geq 0 \ \ \forall x, y, \in X$, and $\|v\| = 0$ iff $v = 0$.

Examples. (1) Let $p \geq 1$. The *p-norm* $\| \ \|_p$ on $\mathbf{R}^n$ is defined by $\|\mathbf{x}\|_p = (\sum_i |x^i|^p)^{1/p}$. Homogeneity and positivity are obvious; the triangle inequality also holds (but would fail for $p < 1$) but is harder to prove. (2) The *sup-norm* $\| \ \|_\infty$ on $\mathbf{R}^n$ is defined by $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x^i|$.

normed vector space. A *normed vector space* is a pair $(V, \| \ \|)$, where V is a vector space and $\| \ \|$ is a norm on V.

Common notational simplification. In statements for which the norm $\| \ \|$ does not need to be referenced explicitly, one often says simply "Let V be a normed vector space" rather than "Let $(V, \| \ \|)$ be a normed vector space."

Normed vector spaces as metric spaces. Let $(V, \| \ \|)$ be normed vector space. The norm induces a metric $d : V \times V \to \mathbf{R}$ defined by $d(x, y) = \|x - y\|$. Whenever metric-space properties of a normed vector space are referred to, it is implicit that the metric is the one induced by the norm unless otherwise stated. (This is a universal convention, not just a convention for these notes).

equivalent norms. Two norms $\| \ \|, \| \ \|'$ on a vector space V are called *equivalent* if there exist $c_1 > 0, c_2 > 0$ such that $c_1\|x\| \leq \|x\|' \leq c_2\|x\|$.

Facts. (1) If two norms on V are equivalent, then so are their induced metrics.

(2) For each $n \geq 0$, all norms on $\mathbf{R}^n$ are equivalent (i.e. any two are equivalent to each other). More generally, all norms on any fixed, finite-dimensional vector space are equivalent. For infinite-dimensional vector spaces, this is very far from true.

boundedness and continuity of linear maps. Let V, W be normed vector spaces. A linear map $T : V \to W$ is *bounded* if there exists c such that $\|T(x)\|_W \leq c\|x\|_V \ \ \forall x \in V$.

Fact. A linear map between normed vector spaces is continuous iff it is bounded.

operator norm. Let V, W be normed vector spaces and $T : V \to W$ a bounded linear map. The *operator norm* of T, induced by the norms on V and W, is defined by $\|T\| := \sup_{\|x\|=1} \|T(x)\|$.

Facts. (1) A linear map is bounded iff its operator norm is finite. The operator norm is simply the infimum of the constants c that work for T in the definition of "bounded" above. (Easy to show.)

(2) Operator norms are *sub-multiplicative*: if T, S are bounded linear maps between normed vector spaces for which the composition $T \circ S$ is defined, then $\|T \circ S\| \leq \|T\|\|S\|$. (Easy to show.)

Banach space. A *Banach space* is a complete normed vector space. *Examples.* (1)

$\mathbf{R}^n$ with any norm. (2) Any Hilbert space. (3) $\mathrm{Hom}^c(V, W)$, the space of bounded linear maps from a Banach space V to a Banach space W, with the operator norm.

1.3 Topological Spaces

topology; open sets. A *topology* on a set X is any collection $\mathcal{U}$ of subsets of X, to be called *open sets* for that topology, satisfying the following conditions:

(i) $\emptyset \in \mathcal{U}$ and $X \in \mathcal{U}$ (the empty set and the whole set are open).

(ii) If $U_1, U_2, \ldots, U_n \in \mathcal{U}$ then $U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{U}$ (the intersection of any *finite* collection of open sets is open)

(iii) If $U_\alpha \in \mathcal{U}$ for all α in some index set A (finite or infinite), then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{U}$ (the union of *any* collection of open sets is open)

topological space. A *topological space* is a pair $(X,\mathcal{U})$, where X is a set and $\mathcal{U}$ is a topology on X. When $\mathcal{U}$ is understood from context, we usually just say "X is a topological space." *Example.* Any metric space.

A topological space $(X,\mathcal{U})$ is called *metrizable* if there exists a metric on X whose collection of open sets is precisely $\mathcal{U}$.

Notation for below. X denotes a topological space with topology $\mathcal{U}$.

relative (or induced) topology. If $Y \subset X$, we define the *relative topology* or *induced topology* on Y by declaring a subset $V \subset Y$ to be open iff $V = Y \bigcap U$ for some open set $U \subset X$. This makes Y a topological space in its own right, a (topological) *subspace* of X.

quotient topology. Let X be a topological space, let $\sim$ be an equivalence relation on X, and let Z be the set of equivalence classes determined by $\sim$. Let $\pi : X \to Z$ be the map that sends an element to its equivalence class. Declare $U \subset Z$ open iff $\pi^{-1}(Z)$ is open in X. It is easy to check that this defines a topology on Z, called the *quotient topology.*

closed set. A subset of X is called *closed* if its complement is open. It follows from the definition of "open" that X is closed, $\emptyset$ is closed, any *finite* union of closed sets is closed, and any intersection (finite or infinite) of closed sets is closed.

open neighborhood; neighborhood. An *open neighborhood* of a point $p \in X$ is an open set containing p. A *neighborhood* of a point $p \in X$ is any set that contains an open neighborhood of p.

punctured (open) neighborhood. Let $p \in X$. A *punctured* neighborhood of p is a set $U - \{p\}$, where U is a neighborhood of p. (Thus a punctured neighborhood of p is not a type of neighborhood of p!) If U is open, we call $U - \{p\}$ a punctured open neighborhood of p.

Hausdorff. X is *Hausdorff* if for all distinct points $p, q \in X$ there are neighborhoods

U, V of p, q respectively, with $U \bigcap V = \emptyset$. *Example*: Any metric space. *Non-example.* The "line with two origins", L, defined as follows. Start with the set $X = \mathbf{R} \times \{\text{horse}, \text{dog}\}$, topologized as two disjoint copies of $\mathbf{R}$. Define an equivalence relation on X by declaring $(x, \text{horse}) \sim (y, \text{dog})$ if and only if $x = y \neq 0$. Define L to be the quotient space of this equivalence relation; i.e. the set of equivalence classes, endowed with the quotient topology. Then every open neighborhood of $(0, \text{horse})$ intersects every open neighborhood of $(0, \text{dog})$, so L is not Hausdorff.

convergence; limit of a sequence. A sequence $\{x_n\}$ in X *converges* to $p \in X$, written $x_n \to p$, if for every open neighborhood U of x, $\exists\ N$ such that $x_n \in U$ for all $n > N$. In this case we also say $\lim_{n\to\infty} x_n = p$ provided that X is Hausdorff. (If X is not Hausdorff, a sequence can converge to more than one point, so in this case we do not use terminology or notation that suggests limits are unique.)

continuous function. A function $f : X \to Y$, where X and Y are topological spaces, is called *continuous* at p if for every neighborhood U of $f(p)$, the inverse image $f^{-1}(U)$ is a neighborhood of p in X (equivalently, if for every open neighborhood U of $f(p)$, the inverse image $f^{-1}(U)$ *contains* an *open* neighborhood of p). If X and Y are metric spaces, this reduces to the usual ϵ-δ definition.

Continuity and limits of sequences. Under "Metric Spaces" above, continuity was defined in terms of limits of sequences. For metric spaces, this definition is equivalent to the preceding one, but for general topological spaces only the following implication is true: If X, Y are topological spaces, $f : X \to Y$ is continuous, and $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y. The converse of this implication (for general topological spaces) is false. However, if X is a metric space and Y is any topological space, then f is continuous at $x \iff$ whenever $x_n \to x$ in X we have $f(x_n) \to f(x)$ in Y.

homeomorphism. A *homeomorphism* between topological spaces is a continuous map with a continuous inverse. Two spaces are *homeomorphic* if there exists a homeomorphism between them.

accumulation point; cluster point. An *accumulation point* or *cluster point* of a sequence $\{x_n\}$ in X is a point $p \in X$ for which every open neighborhood of p is visited infinitely often by the sequence (i.e. for every open neighborhood U of p, there exist infinitely many n for which $x_n \in U$). Equivalently, p is an accumulation point of $\{x_n\}$ if there exists a subsequence of $\{x_n\}$ converging to p. *Example.* $X = \mathbf{R}$, $x_n = (-1)^n(1 - 1/n)$. The sequence clusters at both ± 1.

The terms *accumulation point*, *cluster point*, and *limit point* are also applied to subsets of X. If $U \subset X$ and $p \in X$, then p is a cluster point (or limit point or accumulation point) of U if every *punctured* open neighborhood of p contains a point of U. This definition applies whether or not $p \in U$. *Example.* If U is the open unit disk in $\mathbf{R}^2$, then every point in the *closed* unit disk is an accumulation point of U.

Note that an accumulation point of a sequence need not be an accumulation point of the *range* of the sequence. For example, a constant sequence has an accumulation point, but the range of the sequence does not.

closure of a subset. The *closure* $\overline{U}$ of a subset $U \subset X$ is the union of U and all its accumulation points. *Example*: The closure of the open unit disk in $\mathbf{R}^2$ is the closed unit disk.

Facts. (1) $\overline{U}$ is always closed. (2) $\overline{U}$ is the intersection of all closed subsets of X that contain U (this is often taken to be the *definition* of $\overline{U}$), and in this sense is the smallest closed set containing U. (3) U is closed iff $\overline{U} = U$.

interior points; interior of a subset. A point p in a subset $U \subset X$ is an *interior* point of U if there exists an open nbhd V of p entirely contained in U. $\mathrm{Int}(U)$, the *interior of* U, is the set of all interior points of U.

Facts. (1) $\mathrm{Int}(U)$ is always open. (2) $\mathrm{Int}(U)$ is the union of all open subsets of X contained in U (this is often taken to be the *definition* of $\mathrm{Int}(U)$), and in this sense is the largest open set contained in U. (3) U is open iff $\mathrm{Int}(U)=U$.

boundary points; boundary of a subset. A point $p \in X$ is a *boundary point* of $U \subset X$ if every open neighborhood of p contains both a point of U and a point of the complement U' $(= X - U)$. The *boundary of* U, written ∂U, is the set of all boundary points of U.

Facts. (1) $\partial U = \overline{U} \bigcap \overline{U'}$. This is often taken as the definition of ∂U.
(2) $\overline{U} = \mathrm{Int}(U) \coprod \partial U$.

dense subset. A subset U of X is *dense* if $\overline{U} = X$ (equivalently if $\forall p \in X$, every neighborhood V of p contains a point of U).

topological-space terms applied to subsets. For simplicity, the definitions of topological properties below (compactness, sequential compactness, connectedness, arcwise connectedness) are given for whole topological spaces, but the terms are also used for *subsets* of topological spaces. A subset of $U \subset X$ is said to have one of these properties if, with the relative topology (induced from X), the topological space U has that property. Thus we can talk about compact subsets, connected subsets, etc.

sequential compactness. X is *sequentially compact* if every infinite sequence has a convergent subsequence (equivalently, if every infinite sequence has an accumulation point). *Example.* The closed unit disk in $\mathbf{R}^2$ is sequentially compact; the open unit disk is not.

open covers; finite covers; subcovers. A *cover* of X is a collection $\mathcal{V}$ of subsets of X whose union is X. Equivalently, $\mathcal{V}$ is a cover of X if $\mathcal{V} \subset \mathcal{P}(X)$ and $\bigcup_{V \in \mathcal{V}} V = X$. A cover $\mathcal{V}$ of X is an *open cover* if each $V \in \mathcal{V}$ is an open set in X. The cover is *finite* if $\mathcal{V}$ consists of only finitely many elements (remember that the *elements* of $\mathcal{V}$ are *subsets* of X, not elements of X.) *Example.* The set of intervals $\{(0,1), (1/4, 5/2), (1/2, 3/2), (1, 2)\}$ is a finite open cover of the interval $(0, 2)$.

Understanding "cover" to mean "cover of X", a *subcover* $\mathcal{U}$ of $\mathcal{V}$ is a cover each of whose elements is an element of $\mathcal{V}$ (equivalently, $\mathcal{U} \subset \mathcal{V}$, remembering that $\mathcal{U}$ and

$\mathcal{V}$ are subsets of $\mathcal{P}(X)$. In the example above, $\{(0,1),(1/2,3/2),(1,2)\}$ is a subcover. Subcovers need not be finite.

compactness. X is *compact* if every open cover of X has a finite subcover.

useful facts about compactness and sequential compactness. (1) A metric space is compact iff it is sequentially compact. (2) Heine-Borel Theorem: A subset $U \subset \mathbf{R}^n$ is compact iff it is closed and bounded.

In metric spaces, why bother with the initially mysterious concept *compactness* when it's equivalent to the more tangible *sequential compactness*? One reason is that many proofs are easier using compactness. Also, one often wants to deal with non-metric topological spaces (even if they are metrizable, one may not wish to specify a metric on them).

Sample proof using compactness. **Proposition.** If $f : X \to Y$ is a continuous map from one topological space to another, and X is compact, then the image $f(X)$ is compact. ("The continuous image of a compact set is compact.") **Proof.** Let $\mathcal{V}$ be an open cover of $f(X)$. For each $V \in \mathcal{V}$, the set $f^{-1}(V)$ is open since f is continuous, and the collection $\mathcal{W} = \{f^{-1}(V)\}_{V\in\mathcal{V}}$ covers X. Since X is compact, $\mathcal{W}$ has a finite subcover, say $W_1, \ldots, W_n$, where $W_i = f^{-1}(V_i)$. Then $\{f(W_i)\}$ covers $f(X)$. But $f(W_i) = V_i$, so $\{V_1, \ldots V_n\}$ is a finite subcover of $\mathcal{V}$. ∎

arcwise (or path-) connectedness; path component. X is called *arcwise connected* or *path-connected* if every pair of points in X can be joined by a curve (also called an *arc* or a *path*). More precisely, X is arcwise connected if for all $p, q \in X$, there exists a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$. The *path component* of $p \in X$ consists of all points $q \in X$ that can joined to p by some path.

connectedness. X is *connected* if it cannot be expressed as the disjoint union of two nonempty open subsets. *Example.* A subset Y of $\mathbf{R}^2$ consisting of two disjoint closed disks is not connected. (Remember that when a topological term is used in the context of subsets, the relative topology is implied. In this example, each of the two closed disks is *open in the relative topology on* Y, even though not open in X. Thus Y can be expressed as the disjoint union of two nonempty sets each of which is Y-open, though not X-open.)

Note that if $X = A \coprod B$ then each set is the complement of the other, and hence if A and B are both open, they are also both closed. Therefore an equivalent definition of "connected" is: X is connected if the only subsets that are both open and closed are X and $\emptyset$.

connected components. For $p, q \in X$, write $p \sim q$ iff there exists a connected subset of X containing p and q. Then $\sim$ is an equivalence relation on X. The equivalence classes are called the *connected components* (or simply *components*) of X.

The connected component containing a given point p is the union of all connected subsets of X that contain p. It follows that for each $p \in X$, the connected component C_p of X containing p is the largest connected set containing p, in the following sense: if $S \subset X$ is connected and $p \in S$, then $S \subset C_p$.

Facts. (1) Arcwise connectedness implies connectedness, but the converse is false. (2) An *open* subset of $\mathbf{R}^n$ is connected iff it is arcwise connected. (In fact, if $U \subset \mathbf{R}^n$ is open and connected, then any two points of U can be joined not only by a continuous path, but by a smooth path.) This fact extends to manifolds. (3) Every connected component of X is a closed set. If X has only finitely many connected components, then each component is open as well.

Sample proof using connectedness. **Generalized Intermediate Value-Theorem.** If $f : X \to Y$ is a continuous map from one topological space to another, and X is connected, then the image $f(X)$ is connected. ("The continuous image of a connected set is connected.") **Proof.** Assume $f(X)$ is not connected and let A, B be two (relatively) open nonempty subsets of $f(X)$ with $f(X) = A \coprod B$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are open (since f is continuous), nonempty, disjoint, and $f^{-1}(A) \coprod f^{-1}(B) = X$. Hence X is not connected, a contradiction. Therefore $f(X)$ is connected. $\blacksquare$