**Bump-functions and the locality of Leibnizian linear operators**

Throughout this discussion, $M$ is a manifold of dimension $n > 0$, and $\mathcal{F}(M)$ denotes the algebra of smooth real-valued functions on $M$. For each $p \in M$, the set $\mathcal{F}_p(M)$ of "smooth functions defined on an open neighborhood of $p$" is defined as in class, and $\mathcal{G}_p(M)$ denotes the algebra of smooth germs of real-valued functions at $p$.

**Definition 1.1** Let $f$ be a real-valued function on $M$. The *support of* $f$, denoted $\mathrm{supp}(f)$, is the closure of the set $\{p \in M \mid f(p) \neq 0\}$. If $U \subset M$ and $\mathrm{supp}(f) \subset U$, we say that $f$ is *supported in* $U$; if, in addition, $\mathrm{supp}(f)$ is compact, we say that $f$ is *compactly supported in* $U$

Let $X$ be a vector field on $M$. Then $X$ acts as a Leibnizian linear operator $\mathcal{F}(M) \to \mathcal{F}(M)$, and, for each $p \in M$, $X$ also acts as a Leibnizian linear operator $\mathcal{G}_p(M) \to \mathbf{R}$ (via the action of $X_p$).[1] We may ask whether the action of $X$ on $\mathcal{F}(M)$ determines the action of $X$ on $\mathcal{G}_p(M)$. In other words, can every germ at $p$ be represented by a smooth real-valued function $f$ *defined on all of* $M$?

The answer is yes. The key to showing this is the existence of "smooth bump-functions":

**Lemma 1.2** *Let $p \in M$ and let $U$ be an open neighborhood of $p$. There exist an open neighborhood $V$ of $p$ and a smooth function $\rho : M \to \mathbf{R}$, compactly supported in $U$, with range in $[0, 1]$, such that $\rho$ is identically 1 on $V$.*

Since the function $\rho$ in the lemma is continuous and achieves the values 0 and 1, its range is *exactly* $[0, 1]$, but there is no need to assert this in the definition or in the later proof.

The proof of Lemma 1.2 will be given later in these notes; for now simply assume the lemma.

**Proposition 1.3** *Let $p \in M$ and let $g$ be a smooth germ at $p$. Then there exists $f \in \mathcal{F}(M)$ such that $g$ is the germ of $f$ at $p$. In fact, given any representative $(f_1, U)$ of $g$, there exists such an $f$ that is compactly supported in $U$.*

**Proof**: Let $(f_1, U)$ be such that $g = [(f_1, U)]$, where "[ ]" denotes equivalence-class under the relation that defines "germ at $p$". Let $V$ and $\rho$ be as in Lemma 1.2. Define $f : M \to \mathbf{R}$ by

$$f(p) = \begin{cases} \rho(p) f_1(p) & \text{if } p \in U, \\ 0 & \text{if } p \in M - U. \end{cases} \tag{1.1}$$

[1] A Leibnizian linear map from one algebra (over a given field, in this case $\mathbf{R}$) to another is also called a *derivation*.

Observe that $M = U \bigcup (M \setminus \mathrm{supp}(\rho))$ (since $\mathrm{supp}(\rho) \subset U$), the union of two open sets. Since $f_1$ and $\rho$ are smooth on $U$, so is the product $\rho f_1$, so $f|_U$ is smooth. By definition $f$ is identically 0 on $M \setminus \mathrm{supp}(\rho)$, hence is smooth on this open set as well. Hence $f$ is smooth, and $\mathrm{supp}(f)$ is a closed (hence compact) subset of the compact set $\mathrm{supp}(\rho)$.

Finally, $f|_V = f_1|_V$, so $[(f, M)] = [(f_1, U)]$. ■

**Remark 1.4** The fact that we can take $\rho$ in Lemma 1.2 to have range in $[0, 1]$, and can take $\rho$ in Lemma 1.2 and $f$ in Proposition 1.3 to have compact support as the indicated therein, will not be used in these notes. However, these facts are important for some applications outside these notes.

**Corollary 1.5** *Let $U \subset M$ be a nonempty open subset of $M$, let $f : M \to \mathbf{R}$ be a smooth function that vanishes identically on $U$, and let $L : \mathcal{F}(M) \to \mathcal{F}(M)$ be a Leibnizian linear operator. Then the function $L(f)$ vanishes identically on $U$.*

**Proof**: Let $p \in U$. Let $V$ and $\rho$ be as in Lemma 1.2. Define $g = (1 - \rho)f : M \to \mathbf{R}$. Then $L(f)|_p = L(1 - \rho)|_p f(p) + (1 - \rho(p))L(f)|_p = 0$, since $f(p) = 0$ and $\rho(p) = 1$. Since $p$ was arbitrary, the conclusion follows. ■

**Corollary 1.6** *Let $L : \mathcal{F}(M) \to \mathcal{F}(M)$ be a Leibnizian linear operator and let $p \in M$. Then for all $f \in \mathcal{F}(M)$, the value of $L(f)$ at $p$ depends only on the germ of $f$ at $p$. (I.e. if $f_1, f_2 \in \mathcal{F}(M)$ have the same germ at $p$, then $L(f_1)|_p = L(f_2)|_p$.)*

**Proof**: Let $f_1, f_2 \in \mathcal{F}(M)$ be functions with the same germ at $p$, and let $g = f_2 - f_1$. Then $g$ vanishes identically on some open neighborhood $U$ of $p$. By Corollary 1.5, $L(g)|_p = 0$. Since $L$ is linear, it follows that $L(f_1)|_p = L(f_2)|_p$. ■

**Remark 1.7** What Corollary 1.6 asserts is a *locality principle* for Leibnizian linear operators $L : \mathcal{F}(M) \to \mathcal{F}(M)$: for all $f \in \mathcal{F}(M)$ and $p \in M$, the value of $L(f)$ at $p$ depends only on the behavior of $f$ in an arbitrarily small neighborhood of $p$.

**Lemma 1.8** *Let $X$ be a "set-theoretic vector field" on $M$: a map $M \to TM$, here denoted $p \mapsto X_p$, such that $X_p \in T_pM$ for all $p \in M$. For each $f \in \mathcal{F}(M)$, define the function $X(f) : M \to \mathbf{R}$ by $X(f)|_p = X_p(f)$. If $X(f)$ is smooth for each $f \in \mathcal{F}(M)$, then so is the map $X : M \to TM$, and therefore $X$ is a (smooth) vector field on $M$.*

**Proof**: (Omitted; it's part of a homework problem.) ■

Using the definition of "vector field" given in class, the "(smooth)" in the last line of the above lemma is redundant. But if we relax the smoothness requirement in that definition, we can define "continuous vector field", "$C^k$ vector field", and "smooth ($= C^\infty$) vector field". The convention we're using in class, for simplicity, is that "vector field" means "smooth vector field" unless otherwise specified.

**Corollary 1.9** *Let $L : \mathcal{F}(M) \to \mathcal{F}(M)$ be a Leibnizian linear operator. Then there exists a unique vector field $X$ on $M$ such that $L(f) = X(f)$ for all $f \in \mathcal{F}(M)$.*

**Proof**: Let $p \in M$ and let $g \in \mathcal{G}_p(M)$. By Proposition 1.3, there exists $f \in \mathcal{F}(M)$ representing the germ $g$. Corollary 1.6 implies that the map $X_p : \mathcal{G}_p(M) \to \mathbf{R}$ given by $g \mapsto L(f)|_p$ is well-defined. For $g_1, g_2 \in \mathcal{G}_p(M)$ and $c_1, c_2 \in \mathbf{R}$, if $f_1, f_2 \in \mathcal{F}(M)$ represent $g_1, g_2$ respectively, then $c_1 f_1 + c_2 f_2$ represents $c_1 g_1 + c_2 g_2$. It follows that since $L$ is linear and Leibnizian, so is $X_p$. Thus $X_p \in T_pM$ (under the canonical identification of $T_pM$ with the space of Leibnizian linear functions $\mathcal{G}_p(M) \to \mathbf{R}$).

Hence the map $X : M \to TM$ defined by $p \to X_p$ is a "set-theoretic" vector field on $M$. By construction, for all $f \in \mathcal{F}(M)$ and all $p \in M$ we have $L(f)|_p = X_p(f)$. Since $L(f)$ is smooth for all $f \in \mathcal{F}(M)$, it follows that so is $X(f)$ (the function defined by $p \mapsto X_p(f)$. By Lemma 1.8, it follows that $X$ is a (smooth) vector field on $M$.

Thus $X$ is a vector field with the property that $L(f) = X(f)$ for all $f \in \mathcal{F}(M)$. If $Y$ is a vector field with this property and $p \in M$, then for all $f \in \mathcal{F}(M)$ it follows that $Y_p(f) = L(f)|_p = X_p(f)$. Proposition 1.3 then implies that $Y_p = X_p$. Since $p$ was arbitrary, $Y = X$. ■

Thus, we have a canonical identification

$$\{\text{vector fields on } M\} \longleftrightarrow \{\text{Leibnizian linear operators } \mathcal{F}(M) \to \mathcal{F}(M)\}.$$

Because of this, we say that a map $L : \mathcal{F}(M) \to \mathcal{F}(M)$ "is" a vector field if $L$ is linear and Leibnizian.

We still must prove Lemma 1.2, which we will do in several stages. Below, "$\gamma$" is not meant to remind you of curves; it's simply the letter that comes after $\alpha$ and $\beta$.

**Lemma 1.10** [2] *Let $a, b \in \mathbf{R}$, with $a < b$. There exists a smooth function $\gamma = \gamma_{a,b} : \mathbf{R} \to \mathbf{R}$ such that*

$$\begin{aligned} \gamma(x) &= 1 \text{ for all } x \le a, \\ \gamma(x) &= 0 \text{ for all } x \ge b, \end{aligned}$$

*and $\gamma$ is strictly decreasing on $[a, b]$.*

**Proof**: Define $\alpha : \mathbf{R} \to \mathbf{R}$ by

$$\alpha(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

Then (as the student should check) $\alpha$ is smooth.

Now define $\beta : \mathbf{R} \to \mathbf{R}$ by $\beta(x) = \alpha(x-a)\alpha(b-x)$. Then $\beta$ is smooth, strictly positive on the interval $(a, b)$, and identically zero outside this interval.

[2] Lemma 1.10 and Corollary 1.12, are essentially copied from M. W. Hirsch, *Differential Topology*, Springer-Verlag 1976.

Finally, define $\gamma : \mathbf{R} \to \mathbf{R}$ by

$$\gamma(x) = \frac{\int_x^b \beta(t)dt}{\int_a^b \beta(t)dt}. \tag{1.2}$$

Then (as the student should check), $\gamma$ has the desired property. ■

**Remark 1.11** Functions such as the function $\beta$ in Lemma 1.10 are often called *bump functions* on $\mathbf{R}$.

In these notes, we will make no use of the fact that the function $\gamma_{a,b}$ in Lemma 1.10 is strictly decreasing on $[a, b]$. It's just nice to know that we can choose $\gamma_{a,b}$ to have this property (in addition to the others stated in Lemma 1.10) in case we ever want to use this fact.

**Corollary 1.12** *Let $a, b \in \mathbf{R}$, with $0 < a < b$, and let $x_0 \in \mathbf{R}^n$ (where $n \geq 1$). Then there exists a smooth function $\lambda = \lambda_{a,b,x_0} : \mathbf{R}^n \to [0, 1]$ such that*

$$\begin{array}{rcl} \lambda(x) & = & 1 \text{ if } \|x - x_0\| \leq a, \\ \text{and} & & \\ \lambda(x) & = & 0 \text{ if } \|x - x_0\| \geq b, \end{array}$$

*where $\|\ \|$ is the Euclidean norm on $\mathbf{R}^n$ ($\|x\| = (\sum_i (x^i)^2)^{1/2}$).*

**Proof**: The function $\lambda$ defined by $\lambda(x) = \gamma_{a,b}(\|x - x_0\|)$, where $\gamma_{a,b}$ is as in Lemma 1.10, has all the desired properties (as the student should check). ■

**Remark 1.13** Functions such as the function $\lambda_{a,b,x_0}$ in Lemma 1.10 are often called *cutoff functions* or *bump functions* on $\mathbf{R}^n$. "Cutoff" is most commonly used only when $x_0 = 0$.

**Proof of Lemma 1.2**: Let $(W, \varphi)$ be a chart containing $p$. Since $(W \cap U, \varphi|_{W \cap U})$ is another such chart, without loss of generality we may (and will) assume that $W \subset U$.

Let $x_0 = \varphi(p)$. For $r > 0$ and $x \in \mathbf{R}^n$, let $B_r(x), \overline{B}_r(x)$ denote the open and closed balls of radius $r$, centered at $x$, with respect to the Euclidean metric on $\mathbf{R}^n$. Since $\varphi(W)$ is open in $\mathbf{R}^n$, we may select $r > 0$ such that $B_r(x_0) \subset \varphi(W)$. Let $a = \frac{r}{4}, b = \frac{r}{2}$, and let $\lambda : \mathbf{R}^n \to [0, 1]$ be a function having the properties of $\lambda_{a,b,x_0}$ in Corollary 1.12. Let $V_1 = \varphi^{-1}(B_{r/4}(x_0))$, $V_2 = \varphi^{-1}(B_{r/2}(x_0))$, and $W = M - \overline{V_2}$. Then $V_1$ is an open neighborhood of $p$, the pair $\{U, W\}$ is an open cover of $M$, and $\lambda$ is compactly supported in $\overline{V_2} \subset U$. Define $\rho : M \to \mathbf{R}$ by

$$\rho(q) = \begin{cases} \lambda(\varphi(q)) & \text{if } q \in U, \\ 0 & \text{if } q \in M - U. \end{cases}$$

Then $\rho$ is compactly supported in $U$, has range in $[0, 1]$, and $\rho|_{V_1} \equiv 1$. Furthermore, $\rho|_U = \lambda \circ \varphi$, a smooth function, and $\rho|_W \equiv 0$, also a smooth function. Hence $\rho$ is smooth. ■