**Differential Geometry—MTG 6256—Fall 2021**
**Problem Set 1**
**Due-date: Fri. 11/5/21**

**Required problems (to be handed in): 1, 2, 4ade.** In doing any of these problems or problem-parts, you may assume the results of all earlier problems or problem-parts (optional or required).

**Optional problems**: All the ones that are not required.

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**Even when homework is well written, reading and grading it is very time-consuming and physically difficult for me**. To keep this task from being more burdensome than it intrinsically needs to be:

1. Use *plain, white, unlined, printer paper with no holes.*

2. Make sure your work is *neat* and easy to read. It should either be typed (preferably in LaTeX) or written in pen or **dark** pencil, and there should be **no over-writing** (superimposing new writing on old, *with or without* erasure of the old writing first).

3. If you type your homework, use 12-point font. (LaTex often defaults to 10-point font. To get 12-point font in, say, the "article" document class, the command I use is \documentclass[12pt]{article}.)

4. **Staple** your sheets together in the upper left-hand corner. Any other means of attachment makes more work for me. The staple should be close enough to the corner that when I turn pages, nothing that you've written is obscured. (If you have trouble getting the staple close enough to the corner to achieve this, you haven't left wide enough margins; see below.)

5. If you are writing on both sides of a sheet of paper, do not use paper/ink/pencil combinations for which the writing on one side of the paper shows on the other side (or darkens it).

6. Please use **wide** margins—at least 1.75"—on *all four edges* (left *and* right *and* top *and* bottom). LaTeX preamble commands that will accomplish this in the "article" document class are

   \setlength{\textwidth}{5 in}
   \setlength{\textheight}{7.3 in}
   \setlength{\oddsidemargin}{.75 in}
   \setlength{\topmargin}{0.2 in}

7. Make sure your sentences are **unambiguous**, as well as being correctly punctuated, grammatically correct, and complete.

Below:

- Whenever we refer to an atlas on an already-fixed manifold, we mean mean an atlas within the (explicitly or implicitly) already-fixed maximal atlas.
- Any time a manifold $M$ is constructed by giving an atlas on a set for which a topology is not stated in advance, the topology on $M$ is taken to be the one induced by the atlas.
- I use the words "natural(ly)" and "canonical(ly)" without a formal, mathematically precise, universally applicable definition. Whenever one of these words comes up, it should be clear from context what it means in that context.
- "Map" always means "continuous map".
- "Smooth map of manifolds" means "smooth map from one manifold to another".

1. Let $M$ be a manifold, $U \subset M$ a nonempty open set. Show that an atlas on $M$ naturally gives rise to an atlas on $U$ of the same dimension, hence that $U$ inherits a manifold structure (making $U$ a codimension-zero submanifold of $M$).

2. *Real and complex projective spaces.* If $V$ is a vector space over a field $\mathbf{F}$, the *projectivization* $P(V)$ is defined (as a set) to be the set of one-dimensional vector subspaces of $V$ ("lines through the origin in $V$"). Alternatively, $P(V) = (V \setminus \{0\})/\sim$, where the equivalence relation $\sim$ is defined by $v \sim w \iff v = tw$ for some $t \in \mathbf{F}$.

Let $n \geq 0$ and let $V = \mathbf{F}^{n+1}$, where $\mathbf{F}$ is either $\mathbf{R}$ or $\mathbf{C}$. Let $\pi : V \to P(V)$ be the quotient map, and denote $\pi(v)$ by $[v]$ whenever convenient. For $0 \leq j \leq n$ define

$$\begin{aligned} \tilde{U}_j &= \left\{(x^0, \dots, x^n) \in \mathbf{F}^{n+1} \mid x^j \neq 0\right\} \subset V, \\ U_j &= \pi(\tilde{U}_j) \subset P(V). \end{aligned}$$

Clearly $\{\tilde{U}_j\}_{j=0}^n$ covers $V \setminus \{0\}$, so $\{U_j\}_{j=0}^n$ covers $P(V)$. In view of the equivalence relation defining $P(V)$, the maps $\tilde{\phi}_j : \tilde{U}_j \to \mathbf{F}^n$ defined by

$$\tilde{\phi}_j(x^0, \dots, x^n) = \left(\frac{x^0}{x^j}, \dots, \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, \dots, \frac{x^n}{x^j}\right) \tag{1.1}$$

induce well-defined maps $\phi_j : U_j \to \mathbf{F}^n$,

$$\phi_j([v]) = \tilde{\phi}_j(v).$$

(In (1.1), it is understood that "$\frac{x^0}{x^j}$" is omitted if $j = 0$ and that "$\frac{x^n}{x^j}$" is omitted if $j = n$. Alternative notation for the right-hand side is

$$(\frac{x^0}{x^j}, \dots, \widehat{\frac{x^j}{x^j}}, \dots, \frac{x^n}{x^j}),$$

where the "hat" denotes deletion of the term indexed by $j$.)

(a) Identify the sets $\phi_j(U_j)$ and $\phi_j(U_i \bigcap U_j)$ $(i \neq j)$ explicitly, and compute the overlap-maps $\phi_i \circ \phi_j^{-1}$. (As always, in the overlap-map expression "$\phi_i \circ \phi_j^{-1}$", it is understood that "$\phi_j$" is short-hand for " $\phi_j|_{U_i \cap U_j}$".)

(b) *Real projective space.* Show that $\{(U_i, \phi_i)\}_{i=0}^n$ is a smooth, $n$-dimensional atlas on $\mathbf{R}P^n := P(\mathbf{R}^{n+1})$. Hence $\mathbf{R}P^n$, with the corresponding maximal atlas, is an $n$-dimensional manifold.

Whenever anyone speaks of $\mathbf{R}P^n$ as a manifold, it's implicit that this is the smooth structure.

(c) *Complex projective space.* Any *real* isomorphism from the two-dimensional *real* vector space $\mathbf{C}$ to $\mathbf{R}^2$, such as $z \mapsto (\mathrm{Re}(z), \mathrm{Im}(z))$, induces a real isomorphism $\mathbf{C}^n \to \mathbf{R}^{2n}$ for $n \geq 1$. By composing the chart-maps $\phi_j$ with such an isomorphism, we obtain maps $U_j \to \mathbf{R}^{2n}$. To avoid notational clutter, in this problem we will abuse notation slightly, allowing "$\phi_j$" to stand both for our previously-defined $U_j \to \mathbf{C}^n$ and for the corresponding map $U_j \to \mathbf{R}^{2n}$.

Show that $\{(U_i, \phi_i)\}_{i=0}^n$ is a smooth, $2n$-dimensional atlas on $\mathbf{C}P^n := P(\mathbf{C}^{n+1})$. In the formula for $\phi_i \circ \phi_j^{-1}$, you may treat $n$-tuples of complex numbers as elements of $\mathbf{R}^{2n}$ wherever necessary. Hence $\mathbf{C}P^n$, with the corresponding maximal atlas, is a $2n$-dimensional manifold.

Whenever anyone speaks of $\mathbf{C}P^n$ as a manifold, it's implicit that this is the smooth structure.

**Remark.** There *is* such a thing as a complex manifold, and as you might conjecture, $\mathbf{C}P^n$ is a complex $n$-dimensional manifold. However, the concept is subtler than you might think, and for us, "manifold" will always mean "real manifold" unless otherwise specified.

(d) For $V = \mathbf{R}^{n+1}$ and $V = \mathbf{C}^{n+1}$, show that the topology on $P(V)$ induced by the atlases in parts (b) and (c) is the same as the quotient topology. (In case you need to review the meaning of *quotient topology*, it's in the handout "Point-Set Topology: Glossary and Review" on the class home page.)

(e) Show that $\mathbf{C}P^1$ is diffeomorphic to $S^2$ by explicitly exhibiting a diffeomorphism $F : \mathbf{C}P^1 \to S^2$ that maps $U_0$ to $S^2 \setminus \{\text{north pole}\}$, and maps $U_1$ to $S^2 \setminus \{\text{south pole}\}$.

**Remark.** $\mathbf{C}P^1$ is also called the *Riemann sphere.* As a set, $\mathbf{C}P^1 = U_0 \coprod \{[(0,1)]\}$ ("$\coprod$" means "disjoint union"). In the Riemann sphere, our set $U_0$ is implicitly identified with $\phi_0(U_0) = \mathbf{C}$, and the point $[(0,1)]$ is regarded as "the point at infinity". If you did part (a) correctly, you should find that both overlap maps are given by $z \mapsto \frac{1}{z}$, with domain $\mathbf{C} \setminus \{0\}$.

(f) Show that the quotient map (or *projection*) $\pi : V \setminus \{0\} \to P(V)$ is a smooth map of manifolds in the cases $V = \mathbf{R}^{n+1}$ and $V = \mathbf{C}^{n+1}$.

3. The *Grassmannian* or *Grassmann manifold* $G_k(\mathbf{R}^n)$ $(0 < k < n)$ is a manifold

whose underlying set is the set of $k$-dimensional subspaces of $\mathbf{R}^n$. (This is a generalization of real projective space; $G_1(\mathbf{R}^n) = P(\mathbf{R}^n) = \mathbf{R}P^{n-1}$. Notations for the Grassmannian vary in the literature: some people use the notation "$G_k(\mathbf{R}^n)$" for the set of subspaces of $\mathbf{R}^n$ of *co*dimension $k$. The notations $G_{k,n}$ and $G_{n,k}$ are also used.)

To define a natural atlas on $G_k(\mathbf{R}^n)$, it is convenient to generalize our definition of "$N$-dimensional chart" to allow charts with values in a fixed $N$-dimensional vector space $V$ that we do not require to be (literally) $\mathbf{R}^N$. For any such $V$, we can fix an isomorphism $\iota : V \to \mathbf{R}^N$, and compose any $V$-valued chart-map with $\iota$ to obtain an $\mathbf{R}^N$-valued chart-map. If $f$ : (open set in $V$) $\to$ (open set in $V$) is an overlap map for a pair of $V$-valued chart-maps, then the overlap map for the corresponding $\mathbf{R}^N$-valued chart-maps is $\iota \circ f \circ \iota^{-1}$, which is exactly as (continuously) differentiable as is $f$. In particular, an atlas of $V$-valued charts is smooth if and only if such a corresponding atlas of $\mathbf{R}^N$-valued charts is smooth. Thus, allowing $V$-valued charts does not change the collection of objects we call smooth $N$-dimensional manifolds.

With this in mind, the "standard" smooth atlas on $G_k(\mathbf{R}^n)$ is constructed as follows. Endow $\mathbf{R}^n$ with the standard inner product. Observe that given any $k$-plane $X$ through the origin, any sufficiently close $k$-plane $Y$ through the origin is the "orthogonal graph" of a unique linear map $T : X \to X^\perp$, where $X^\perp$ is the orthogonal complement of $X$. ("Sufficiently close" means that $Y \bigcap X^\perp = \{0\}$. "Orthogonal graph of $T$" means $\{v + T(v) \mid v \in X\}$. Since $\mathbf{R}^n = X \oplus X^\perp$, there is a natural bijection between $\mathbf{R}^n$ and $X \times X^\perp$. Composing appropriately with this bijection identifies the orthogonal graph of $T$ with the "true" graph of $T$.) For each $k$-element subset $I = \{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$, let $X_I$ be the subspace consisting of all $x \in \mathbf{R}^n$ all of whose coordinates other than those in positions $i_1, \ldots, i_k$ vanish. Let $V_I \subset \mathbf{R}^n$ be the (set-theoretic) complement of $X_I^\perp$ in $\mathbf{R}^n$.

(a) Let $\mathcal{I} = \{(i_1, i_2, \ldots, i_k) \in \mathbf{N}^k : 1 \le i_1 < i_2 < \cdots < i_k \le n\}$. Show that $\{V_I\}_{I \in \mathcal{I}}$ is an open cover of $\mathbf{R}^n \setminus \{0\}$ and determines a cover $\{U_I\}$ of $G_k(\mathbf{R}^n)$, that, for $k = 1$, reduces to the cover used in problem 2b (modulo replacing $\mathbf{R}^{n+1}$ with $\mathbf{R}^n$).

(b) Show that there is a natural bijection from $U_I$ to $\mathrm{Hom}(X_I, X_I^\perp)$, a space naturally isomorphic to $M_{(n-k)\times k}(\mathbf{R})$ (the space of $(n-k) \times k$ real matrices). Hence there is a natural bijection $\phi_I : U_I \to M_{(n-k)\times k}(\mathbf{R})$.

(c) Show that the overlap maps $\phi_J \circ \phi_I^{-1}$ are smooth (this requires quite a bit more work for general $k$ than did the $k = 1$ case in problem 2), and hence that $G_k(\mathbf{R}^n)$ is a smooth manifold of dimension $k(n-k)$.

4. *Covering spaces.* A *covering space* of topological space $X$ is a pair $(\widetilde{X}, \pi)$, where $\widetilde{X}$ is a topological space and $\pi : \widetilde{X} \to X$ is a continuous surjective map with the following property: for each $p \in X$, there is an open neighborhood $U$ of $p$ that is *evenly covered*, meaning that $\pi^{-1}(U)$ is a union of disjoint open sets in $\widetilde{X}$, each of which is mapped homeomorphically by $\pi$ to $U$. (Surjectivity is automatic if "union" is replaced by "non-empty union".) The map $\pi$ is called the *projection* or the *covering*

*map.*

Below, assume that $M, N$ are manifolds and that $(\widetilde{M}, \pi)$, $(\tilde{N}, \pi')$ are covering spaces of $M, N$ respectively.

(a) Let $m = \dim(M)$. Show that an atlas on $M$ gives rise to an $m$-dimensional atlas on $\widetilde{M}$, hence that $\widetilde{M}$ naturally inherits the structure of a smooth $m$-dimensional manifold. (For this reason we usually refer to $\widetilde{M}$ or $(\widetilde{M}, \pi)$ as a covering *manifold* of $M$, rather than just a covering *space*.)

(b) **Definition.** Let $X, Y$ be manifolds. A map $F : X \to Y$ is a *local diffeomorphism* if $F$ is an open map and every $p \in X$ has an open neighborhood $U$ with the property that $F|_U : U \to F(U)$ is a diffeomorphism. Here, $F(U)$ is an open set since the map $F$ is open, and naturally carries a manifold structure by problem 1. (In part (f) of this problem, we will see that this definition is equivalent to a briefer, more standard, but less self-explanatory one.)

Show that the natural smooth structure (equivalently, maximal atlas) on $\widetilde{M}$ in part (a) is the unique smooth structure for which $\pi$ is a local diffeomorphism.

(c) Suppose that $\widetilde{M}$ is a manifold of dimension $m$. Assume that for any two open sets $\widetilde{U}_1, \widetilde{U}_2 \subset \widetilde{M}$ for which $\pi|_{\widetilde{U}_i}$ is injective and for which $\pi(\widetilde{U}_1) = \pi(\widetilde{U}_2)$, the map $(\pi|_{\widetilde{U}_2})^{-1} \circ \pi|_{\widetilde{U}_1}$ is smooth. (Hence all such maps are diffeomorphisms.) Show that $M$ naturally inherits the structure of a smooth $m$-dimensional manifold.

(d) Let $F : M \to N$ be a continuous map, and let $\tilde{F}$ be a continuous map from $\widetilde{M}$ to $N$, from $M \to \tilde{N}$, or from $\widetilde{M}$ to $\tilde{N}$. If the corresponding diagram in Figure 1 commutes, we call $\tilde{F}$ a *lift* of $F$. (Figure 1 is somewhere nearby, wherever LaTeX felt like putting it.) Show that for lifts of all three types, cases, if $\tilde{F}$ is a lift of $F$, then $\tilde{F}$ is smooth if and only if $F$ is smooth.

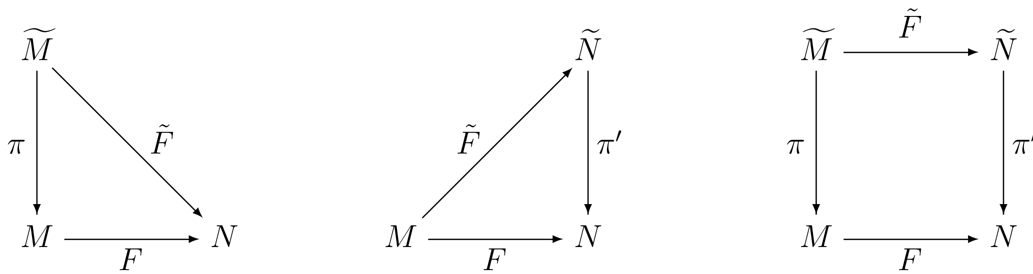


Figure 1: Diagrams for problem 4

*Note*: Given only $F : M \to N$, a unique lift $\tilde{F} : \widetilde{M} \to N$ always exists, namely $\tilde{F} := F \circ \pi : M \to N$. The other two types of lifts do not always exist, and when they exist, they may not be unique. Given only $\tilde{F}$ (of any of the three types indicated with this notation above), we say that $\tilde{F}$ *descends* to a map $M \to N$ if there exists $F : M \to N$ of which $\tilde{F}$ is a lift. A map $\tilde{F} : M \to \tilde{N}$ descends uniquely to the map

$F := \pi' \circ \tilde{F}$; in the other two cases, $\tilde{F}$ does not always descend, but when it does descend, the map to which it descends is unique.

(e) For smooth maps between manifolds $X$ and $Y$, the customary definition of *local diffeomorphism is*: a smooth map $F : X \to Y$ is a local diffeomorphism if, for every $p \in X$, the derivative $F_{*p} : T_pX \to T_{F(p)}Y$ is an isomorphism. Show that this definition is equivalent to the one in part (b).