Differential Geometry—MTG 6256—Fall 2021
Problem Set 2
Due-date: Wednesday 12/8/21

Required problems: 3, 4, 7, 8 (a short answer suffices for #8, but be as precise as you can), **12, 13**. For problems 12–16 and beyond, there is a 4-page set of notes to read first; see instructions on pp. 4–5.

In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Required reading: The statements of all the problems and Remarks. (Consider the Remarks to be part of one of the missed lectures.) However, since this is a lot of reading, the due-date for reading the optional problems and remarks is the start of the spring semester.

Optional problems: All the ones that are not required.

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1. Parts (a), (b), (c) of this problem have no dependence on each other; they can be done in any order.

Let $F : M \to N$ be a smooth map of manifolds.

(a) Prove that if $X \subset M$ is a submanifold, then $F|_X : X \to N$ (the restriction of F to X) is also a smooth map of manifolds.

(b) Suppose that the image of F contained in a submanifold Y of N. Prove that F, viewed as a map $M \to Y$, is also a smooth map of manifolds.

(c) Define $G : M \to M \times N$ by $G(p) = (p, F(p))$. Show that G is a smooth map of manifolds, and that its image is a submanifold of $M \times N$.

Remark: Let us informally define "smooth subset of a manifold" to mean "submanifold". The *image* of a smooth map need be smooth (for example, consider the image of the sine function $\mathbf{R} \to \mathbf{R}$), but what you've shown above is that the *graph* of a smooth map is always smooth (as, indeed, the graph of the sine function is).

2. **Hopf maps**. For $V = \mathbf{C}^{n+1} \cong_{\mathbf{R}} \mathbf{R}^{2n+2}$, let H be the restriction of the projection π to the unit sphere $S^{2n+1} \subset \mathbf{R}^{2n+2}$.

(a) Show that H is surjective and smooth. (Note: there is a reason that one part of problem 1 was given before this problem.)

(b) For $n = 1$, let $F : \mathbf{C}P^1 \to S^2$ be the diffeomorphism you found in problem 2(e) of Problem Set 1, and find an explicit formula for $H \circ F : (S^3 \subset \mathbf{C}^2) \to S^2$. The name "the Hopf map" is used for both the map $H : S^3 \to S^2$ and the composition $H \circ (\text{diffeo } \mathbf{C}P^1 \to S^2)$. For $n > 1$, the maps $H : S^{2n+1} \to \mathbf{C}P^n$ are called *generalized Hopf maps*. (For $n > 1$, the space $\mathbf{C}P^n$ is *not* homeomorphic [let alone diffeomorphic] to a sphere!)

3. Let $\mathrm{Sym}_n(\mathbf{R}) \subset M_{n\times n}(\mathbf{R}) \cong \mathbf{R}^{n^2}$ be the subspace consisting of $n \times n$ symmetric matrices (those A for which $A^t = A$). Define $F : M_{n\times n}(\mathbf{R}) \to \mathrm{Sym}_n(\mathbf{R})$ by $F(A) = A^t A$. Let $I \in M_{n\times n}(\mathbf{R})$ be the identity; note that $F^{-1}(I) = \{A \in M_{n\times n}(\mathbf{R}) \mid A^t A = I\}$, which is also known as the *orthogonal group* $O(n)$. (The word "group" applies since $O(n)$ is a group under matrix-multiplication.) Show that I is a regular value of F, and hence that $O(n)$ is a submanifold of $M_{n\times n}(\mathbf{R})$. What is the dimension of $O(n)$?

Note: $O(n)$ is not connected; it has two connected components, the set $SO(n)$ of orthogonal matrices of determinant 1, and the set of orthogonal matrices of determinant -1. (It takes a little work to show that $SO(n)$ is connected.) This example illustrates the fact that non-connected manifolds can arise naturally in important examples.

4. Let M, N be manifolds, with M compact and N connected. Show that if $F : M \to N$ is a submersion, then F is surjective.

5. Let M, N be manifolds *of equal dimension*, with M compact and N connected. Prove that if M can be embedded in N, then M and N are diffeomorphic. (Thus, for example, the sphere S^2 cannot be embedded in the torus, or vice-versa.)

Note: We saw in class that the conclusion "M and N are diffeomorphic" would be false if we removed either the hypothesis that M is compact or the hypothesis that N is connected.

6. Let M, N be manifolds and let $F : N \to M$ be an embedding. Show that $F(N)$ is a submanifold of M. (A sketch of this argument was given in class.)

7. Let $F : M \to N$ be a smooth map of manifolds, let $q \in \mathrm{image}(F)$, and assume that q is a regular value of F. Then, by the Regular Value Theorem, $Z := F^{-1}(q)$ is a submanifold of M. Let $j : Z \to M$ be the inclusion map, and let $p \in Z$. Show that

$$j_{*p}(T_p Z) = \ker(F_{*p}).$$

(In words: the tangent space at p to the *fiber containing* p—i.e. the set $F^{-1}(F(p))$—is the kernel of derivative of F at p.)

8. Let M and N be manifolds of dimensions m and n respectively. For $p \in M$, $q \in N$, how is $T_{(p,q)}(M \times N)$ related to $T_p M$ and $T_q N$?

9. **Transversality.** This optional multi-part problem deals with a very important concept and tool in differential topology, and the last part of it is essential to the differential-topological definition and interpretation of the *degree* of a smooth map from one compact n-dimensional manifold to another.

Notation: Given two vector subspaces U, V of a vector space W, we define their *sum* $U+V$ to be the subspace $\{u+v \mid u \in U, v \in V\}$ (also called span$\{U,V\}$).[1]

Two submanifolds M and Z of a manifold N are said to intersect *transversely* at a point $z \in N$ if $T_zM + T_zZ = T_zN$ (more precisely, if $\iota_{*z}(T_zM) + j_{*z}(T_zZ) = T_zN$, where ι, j are the inclusion maps of M, Z, respectively, into N). If this condition is met at all points of $M \bigcap Z$ we say simply that M and Z *intersect transversely*, or have *transverse intersection*, or that *the intersection is transverse*, and write $M \pitchfork Z$.

More generally, given manifolds M, N and a submanifold $Z \subset N$, a map $F : M \to N$ is said to be *transverse to* Z if for all $(p,z) \in M \times Z$ with $F(p) = z$, we have $F_{*p}(T_pM) + T_zZ = T_zN$. Short-hand notation for "F is transverse to Z" is "$F \pitchfork Z$". We may view this as a generalization of the definition in the previous paragraph, since in the case of two submanifolds M, Z of N, the submanifolds intersect transversely if and only if the inclusion map $\iota : M \to N$ is transverse to Z. (It's clear that this relation is symmetric in M, Z.) Note that in this case, $\iota^{-1}(Z) = M \bigcap Z$.

Transversality comes into play when we ask the question "Is the intersection of two submanifolds a submanifold?" The answer is no in general, but yes if the intersection is transverse. Transversality is a *sufficient*, but not necessary, condition for the intersection to be a submanifold. Some examples with $N = \mathbf{R}^3$, with coordinates x, y, z: (i) the submanifolds $Z = xy$-plane, $M = yz$-plane, intersect transversely; (ii) $Z = xy$-plane, $M = z$-axis, intersect transversely; (iii) $Z = x$-axis, $M = y$-axis, do not intersect transversely ; (iv) $Z = xy$-plane, $M = \{\text{graph of } z = x^2 - y^2\}$, do not intersect transversely (because of what happens at the origin).

(a) Let $N = \mathbf{R}^n$, $0 \le k \le n$, and view N as $\mathbf{R}^k \times \mathbf{R}^{n-k}$. (For the cases $k = 0$ and $k = n$, the convention is $\mathbf{R}^0 = \{0\}$ and we make the obvious identifications of $\{0\} \times \mathbf{R}^n$ and $\mathbf{R}^n \times \{0\}$ with $\mathbf{R}^n$.) Let Z be the k-dimensional submanifold $\mathbf{R}^k \times \{0 \in \mathbf{R}^{n-k}\}$. Prove that if M is a manifold and $F : M \to N$ is transverse to Z, then $F^{-1}(Z)$ is a submanifold of M. (*Hint*: Consider the map $G = \pi \circ F : M \to \mathbf{R}^{n-k}$, where $\pi : \mathbf{R}^k \times \mathbf{R}^{n-k} \to \mathbf{R}^{n-k}$ is projection onto the second factor.)

(b) Use the result of part (a) to prove that if M, N are arbitrary manifolds and $F : M \to N$ is transverse to a submanifold $Z \subset N$, then $F^{-1}(Z)$ is a submanifold of M. (Note that the case $Z = \{\text{point}\}$ is the Regular Value Theorem, so the theorem you're asked to prove here may be considered a generalization.) What are the dimension and codimension of $F^{-1}(Z)$?

(c) Part (b), applied to the case in which F is the inclusion map of a submanifold $M \subset N$, shows that if $M \pitchfork Z$ and $M \bigcap Z \neq \emptyset$, then $M \bigcap Z$ is a submanifold of M. For $p \in M \bigcap Z$, express $T_p(M \bigcap Z)$ in terms of T_pM and T_pZ.

(d) In the setting of part (c), $M \bigcap Z$ is also a submanifold of Z, by symmetry. It is easy to show that a submanifold of a submanifold of N is a submanifold of N, so:

[1]Note that U and V are allowed to have nontrivial intersection. When the intersection is trivial, i.e. $U \bigcap V = \{0\}$, we say that W is the *direct* sum of U and V, and (sometimes) write $W = U \oplus V$. However, we also use the symbol "$\oplus$" for the direct sum of two arbitrary vector spaces that aren't given to us as subspaces of a third.

- M is a submanifold of N, of a certain codimension;
- Z is a submanifold of N, of a certain codimension;
- $M \bigcap Z$ is a submanifold of M, of a certain codimension;
- $M \bigcap Z$ is a submanifold of Z, of a certain codimension; and
- $M \bigcap Z$ is a submanifold of N, of a certain codimension.

Express the last three codimensions on this list in terms of the first two. To understand what these relations are saying, after you figure out the formulas, write them out without choosing letters to represent dimensions or codimensions; i.e. using the terms "codimension of M in N", "codimension of $M \bigcap Z$ in M", etc. Try to formulate a general principle that explains (not necessarily rigorously) your findings.

(e) Independent of the earlier parts of this problem, what is a necessary and sufficient condition that a subset S of a given manifold be a zero-dimensional submanifold? (The condition should involve nothing more than point-set topology.) Apply this condition when M, Z are transversely-intersecting submanifolds of N of complementary dimensions ($\dim(M) + \dim(Z) = \dim(N)$). What do you conclude about $M \bigcap Z$ in this case? If both M and Z are compact, what stronger conclusion can you reach?

10. Let M be a manifold and let $f, g : M \to \mathbf{R}$ be smooth functions. Show that

$$d(fg) = g\,df + f\,dg.$$

11. Let X be a "set-theoretic" vector field on a manifold M (a map $p \to X_p$ from M to TM for which $X_p \in T_pM$ for all $p \in M$ but for which no smoothness, or even continuity, is required). Show that the following are equivalent:

(i) Viewed as a map $M \to TM$, the map X is smooth.

(ii) For every chart (U, φ), with associated local coordinates $\{x^i\}_{i=1}^n$, the functions $X^i : U \to \mathbf{R}$ defined pointwise by $X|_U = \sum_i X^i \frac{\partial}{\partial x^i}$ are smooth.

(iii) For all (nonempty) open sets $U \subset M$ and smooth functions $f : U \to \mathbf{R}$, the function $X(f) : U \to \mathbf{R}$ is smooth.

Instructions for problems 12–16

Before starting problems 12–16, read the notes "Bump-functions and the locality of Leibnizian linear operators" posted on the class home page. Some of the problems require facts proven in these notes. Corollary 1.9 in these notes makes use of problem 11 above, but there is no circular reasoning.

In problem 12, given a manifold M:

- $\mathcal{F}(M)$ denotes the algebra of smooth functions $M \to \mathbf{R}$.
- $\mathrm{Leib}(M)$ denotes the space of Leibnizian linear maps $\mathcal{F}(M) \to \mathcal{F}(M)$.
- For $p \in M$,
  - $\mathcal{F}_p(M) = \{(f, U) \mid U$ is an open neighborhood of p and $f : U \to \mathbf{R}$ is smooth$\}$.
  - $\mathcal{G}_p(M)$ denotes the algebra of germs at p of smooth real-valued functions (the quotient of $\mathcal{F}_p(M)$ by the equivalence relation "$(f_1, U_1) \sim (f_2, U_2)$ iff $U_1 \cap U_2$ contains an open nbhd U of p such that $f_1|_U = f_2|_U$".
  - $\mathrm{Leib}_p(M)$ denotes the space of Leibnizian linear maps $\mathcal{F}(M) \to \mathbf{R}$.
  - $\mathrm{Leib}_p^{\mathcal{G}}(M)$ denotes the space of Leibnizian linear maps $\mathcal{G}_p(M) \to \mathbf{R}$.

Remark: A Leibnizian linear map from one algebra (over a given field, in this case $\mathbf{R}$) to another is also called a *derivation*. The term "a derivation *on* [or *of*] an algebra" is usually reserved for a derivation from an algebra to itself.

Note that for any nonempty set S and vector space V, the set $\mathrm{Func}(S, V)$ of *all* functions $S \to V$ inherits the structure of a vector space (via pointwise operations). It is easily seen that $\mathrm{Leib}(M)$ and $\mathrm{Leib}_p(M)$ are vector subspaces of $\mathrm{Func}(\mathcal{F}(M), \mathcal{F}(M))$ and $\mathrm{Func}(\mathcal{F}(M), \mathbf{R})$, respectively, hence are vector spaces (canonically). This is the meaning of "space" in "space of Leibnizian linear maps".

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12. Let M be a manifold and let $p \in M$. Show that there is a canonical isomorphism $\mathrm{Leib}_p^{\mathcal{G}}(M) \to \mathrm{Leib}_p(M)$.

Remark. Since we have previously exhibited a canonical isomorphism $T_pM \to \mathrm{Leib}_p^{\mathcal{G}}(M)$, we therefore have a canonical isomorphism $T_pM \to \mathrm{Leib}_p(M)$. Thus, instead of regarding a vector $v \in T_pM$ as an operator on germs, or as a map $\mathcal{F}_p(M) \to \mathbf{R}$ that determines such an operator, we can regard v simply as an operator on $\mathcal{F}(M)$ (real-valued, linear, and Leibnizian, of course). This fact affords some convenience, since it is simpler to say "Let f be a smooth function $M \to \mathbf{R}$" than to say "Let U be an open neighborhood of p and let $f : U \to \mathbf{R}$ be a smooth function," or to say "Let g be a smooth germ at p and let (f, U) be a representative of g." This fact is used in many definitions and proofs involving tangent vectors and/or vector fields. However, it is still important to remember that for $v \in T_pM$ and $f \in \mathcal{F}(M)$, the value of $v(f)$ depends only on the germ of f at p.

13. Let M be a manifold, let $p \in M$, and let $v \in T_pM$. Show that there exists a vector field X on M with $X_p = v$. (In other words, every tangent vector at a point can be extended to a vector field on M.)

14. Let M be a manifold, let $p \in M$, and let $\xi \in T_p^*M$. Prove that there exists a smooth function $f : M \to \mathbf{R}$ such that $f(p) = 0$ and $df|_p = \xi$.

15. Let M be a manifold and let $\pi : TM \to M$ be the natural projection. Show in the following two ways that π is a submersion: (a) by using charts $(\tilde{U}, \tilde{\varphi})$ of TM induced by charts (U, φ) of M as discussed in class; (b) using problem 13, the Chain Rule for maps of manifolds, and simple linear algebra. (*Hint*: View a vector field X as a map $X : M \to TM$ satisfying $\pi \circ X =$ identity.) You may substitute a different method for part (a) if you've figured out another proof (different from the one in (b)).

16. Let M be a manifold. Show that every vector field X on M, viewed as a map $M \to TM$, is an embedding. (The hint given in problem 13 is useful here too.)

Remark 1. It follows that the image of X is a submanifold of TM diffeomorphic to M. Note that every manifold has a canonical vector field, namely the zero vector field. Both this vector field and the corresponding submanifold of TM are called the *zero-section*.

Remark 2. In case you know what a vector bundle is: problems 13–15 and Remark 1 generalize easily to any vector bundle.

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17. Recall that a *projection* from a vector space V to itself is a linear map $P : V \to V$ such that $P^2 = P$.

Let V be a vector space of finite dimension n, assume $0 < k \le n$, and define a linear map $Q : V^{\otimes k} := \overbrace{V \otimes \ldots \otimes V}^{k \text{ factors}} \to \bigwedge^k(V)$ by setting

$$Q(v_1 \otimes \ldots \otimes v_k) = v_1 \wedge \cdots \wedge v_k$$

and extending linearly. Show that $Q = c(k)P$ for some projection $P : V^{\otimes k} \to \bigwedge^k(V) \subset V^{\otimes k}$ and scalar $c(k)$, and give the value of $c(k)$.

18. Let V be a finite-dimensional vector space. For all $j, k \ge 0$, we have defined the wedge-product map $\bigwedge^j V^* \times \bigwedge^k V^* \to \bigwedge^{j+k} V^*$, $(\omega, \eta) \mapsto \omega \wedge \eta$. Thus we have actually defined a *collection* of wedge-product maps, indexed by pairs (j, k) of non-negative integers.

(a) Show that this collection of maps is associative, in the following sense: for all $j, k, l \ge 0$ and $\omega \in \bigwedge^j V^*, \eta \in \bigwedge^k V^*, \xi \in \bigwedge^l V^*$,

$$(\alpha \wedge \beta) \wedge \xi = \alpha \wedge (\beta \wedge \xi). \tag{0.1}$$

(b) **Set-up**: As noted in class, careful wording had to be used in part (a), because, by definition, the only operations that can be *associative* are binary operations *on a single set* S, i.e. maps $S \times S \to S$. What equation (0.1) actually says, in temporary, self-explanatory notation that we'll use for the rest of this problem, is

$$(\alpha \wedge_{j,k} \beta) \wedge_{j+k,l} \xi = \alpha \wedge_{j,k+l} (\beta \wedge_{k,l} \xi). \tag{0.2}$$

There are two ways we can modify our definition of wedge-product so that "$\wedge$" becomes a true associative operation on some set S. The naive way is to take the underlying set for the operation to be $S = \coprod_k \bigwedge^k V^*$. Then, for any $\omega, \eta \in S$, we have $\omega \in \bigwedge^j V^*, \eta \in \bigwedge^k V^*$ for some unique j, k, and we can make definition $\omega \wedge \eta = \omega \wedge_{j,k} \eta$. (The last equation was used implicitly when we defined wedge-product in class.) With this definition, $\wedge$ becomes a map $S \times S \to S$, and part (a) shows that this binary operation is associative.

As discussed briefly in class, a more elegant and useful (if initially less intuitive) solution is to define wedge-product as an operation on the *direct sum* of the vector spaces $\bigwedge^k V^*$, rather than on the *disjoint union*. Specifically, define $\bigwedge^*(V^*) = \bigoplus_{k \geq 0} \bigwedge^k V^*$. (The star in "$\bigwedge^*$" has nothing to do with pullback or dualization; it's just a place-holder for the degrees in the direct summands.) Rather than writing an element of the direct sum in the form $(\omega_0, \omega_1, \omega_2, \dots)$, with $\omega_k \in \bigwedge^k V^*$, it is convenient to use the canonical identification of $\bigwedge^k V^*$ with a subspace of the direct sum (the subspace whose point-set is $\{0\} \times \{0\} \times \cdots \times \{0\} \times \bigwedge^k V^* \times \{0\} \times \{0\}$), allowing us to write $(\omega_0, \omega_1, \omega_2, \dots)$ as $\sum_k \omega_k := \omega_0 + \omega_1 + \omega_2 + \dots$ (well-defined since there are only finitely many nonzero terms in the sum). We then define

$$\left(\sum_k \omega_k\right) \wedge \left(\sum_k \eta_k\right) = \sum_{k,l} \omega_k \wedge_{k,l} \eta_l \tag{0.3}$$

Problem: Show that the map $\wedge : \bigwedge^* V^* \times \bigwedge^* V^* \to \bigwedge^* V^*$ defined by (0.3) is bilinear and associative.

Remark. For both the "naive" and "elegant" ways of defining approach to defining wedge-product as a binary operation $\wedge$ on some set, the restriction of $\wedge$ to $\bigwedge^j V^* \times \bigwedge^k V^*$ is precisely the map $\wedge_{j,k}$, which allows us to drop the subscripts (and write (0.2) as (0.1)) without abusing notation. However, with the naive definition, "bilinear" is meaningless, since the disjoint union of vector spaces is not a vector space. The property of bilinearity is an important feature of wedge-product. Although each of the maps $\wedge_{j,k}$ is bilinear, if we want to have a true associative operation "$\wedge$" that is also bilinear, we must use the second approach.

In class, we previously mentioned the second approach in defining the tensor algebra of a vector space. The idea is very general and important, and is encapsulated in the concept of a $\mathbf{Z}$-*graded algebra*.

A $\mathbf{Z}$–*graded vector space* is a vector space of the form $W_* := \bigoplus_{k \in \mathbf{Z}} W_k$, where $\{W_k\}_{k \in \mathbf{Z}}$ is a collection of vector spaces indexed ("graded") by $\mathbf{Z}$. For simplicity, canonically identify W_k with its image in the direct sum. A $\mathbf{Z}$-*graded algebra* is a $\mathbf{Z}$-graded vector space W_* equipped with a bilinear map $\star : W_* \times W_* \to W_*$ that, for each index-pair (j, k), maps $W_j \times W_k$ into W_{j+k}.

More generally, given all the data above but with the index-set $\mathbf{Z}$ replaced by any nonempty subset A of $\mathbf{Z}$ closed under addition (setting $W_* = \bigoplus_{k \in A} W_k$), we can define a $\mathbf{Z}$-graded algebra $(\tilde{W}_*, \tilde{\star})$ by (i) defining $\tilde{W}_k = W_k$ for $k \in A$, and $\{0\}$ for $k \notin A$, (ii) setting $\tilde{W}_* = \bigoplus_{k \in \mathbf{Z}} \tilde{W}_k$ (which we may canonically identify with $W_* \oplus W'_*$, where $W'_* = \bigoplus_{k \notin A} \tilde{W}_k \cong \{0\}$), and (iii) defining $w \tilde{\star} v = w \star v$ if $w, v \in W_*$, and

$w \tilde{\star} v = 0$ otherwise. The inclusion map of W_* in $\tilde{W}_*$ is a graded-algebra isomorphism: an isomorphism of $\mathbf{Z}$-graded algebras that preserves grading. Since this identification of $(W_*, \star)$ with a $\mathbf{Z}$-graded algebra is canonical, we allow ourselves to call $(W_*, \star)$ itself a $\mathbf{Z}$-graded algebra. In particular, we do this often when A consists of the non-negative integers.

If the data we start with are just a collection $\{W_k\}_{k \in A \subset \mathbf{Z}}$ of vector spaces (with A nonempty and closed under addition) and a collection of bilinear maps $\star_{j,k} : W_j \times W_k \to W_{j+k}$ (one map for each index-pair (j,k)), we canonically construct a $\mathbf{Z}$-graded algebra by setting $W_* = \bigoplus_{k \in A} W_k$ and defining $\star : W_* \times W_* \to W_*$ by $\left(\sum_j w_j\right) \star \left(\sum_j v_j\right) = \sum_{j,k} w_j \star_{j,k} v_k$. (Remember that by definition of "direct sum of an arbitrary collection of vector spaces", there are only finitely many nonzero terms in $\sum_j w_j$ and $\sum_j v_j$, so the sums in this equation are well-defined.) It is easily seen that $\star$ is bilinear and maps $W_j \times W_k$ into W_{j+k} for all $j, k \in A$. Hence $(W_*, \star)$ is a $\mathbf{Z}$-graded algebra. If the collection of maps $\star_{j,k}$ is "associative in the sense of problem 5a", then $\star$ is (truly) associative, and $(W_*, \star)$ is an associative algebra.

All of the above works with the index-set $\mathbf{Z}$ replaced by any abelian group G, so we may speak of G-graded algebras. The grading-groups that arise most often are $\mathbf{Z}$ and $\mathbf{Z}_2$.

19. Let M, N, Z be manifolds. Recall that given a smooth map $F : M \to N$ and a k-form ω on N, with $k > 0$, the *pullback of* ω *by* F is the k-form $F^*\omega$ on M defined by

$$(F^*\omega)_p(v_1, \ldots, v_k) = \omega_{F(p)}(F_{*p}\, v_1, \ldots, F_{*p}\, v_k) \quad \forall p \in M \text{ and } v_1, \ldots, v_k \in T_pM. \tag{0.4}$$

(a) Let $F : M \to N$ be a smooth map and let $k \geq 0$. Show that the map $\Omega^k(N) \to \Omega^k(M)$ given by $\omega \mapsto F^*\omega$ is linear.

(b) Let $F : M \to N$ be a smooth map and let ω, η be differential forms on N. Show that $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$. (Do not forget the case in which the degree of ω or η is zero.)

(c) Let $F : M \to N$ and $G : N \to Z$ be smooth maps, and let ω be a differential form (of arbitrary degree) on Z. Show that $(G \circ F)^*\omega = F^*(G^*\omega)$.

(d) Show that if $F : M \to N$ is a diffeomorphism, then the linear map $F^* : \Omega^k(N) \to \Omega^k(M)$ is invertible for each k, with inverse given by $(F^*)^{-1}\eta = (F^{-1})^*\eta$.

Remark 1. Using the same idea as in problem 18 and the subsequent Remark, we define $\Omega^*(M) = \bigoplus_{k \geq 0} \Omega^k(M)$, and use the collection of wedge-product maps $\Omega^j(M) \times \Omega^k(M) \to \Omega^{j+k}(M)$ to define wedge-product as a bilinear map $\Omega^*(M) \times \Omega^*(M) \to \Omega^*(M)$. The space $\Omega^*(M)$, equipped with this wedge-product, is an associative $\mathbf{Z}$-graded algebra. Parts (a) and (b), combined, are therefore equivalent to the simple statement that, for any smooth map $F : M \to N$, the map $F^* : \Omega^*(N) \to \Omega^*(M)$ is a graded-algebra isomorphism.

Remark 2. Given finite-dimensional vector spaces V, W, and a linear map $L : V \to W$, the *natural adjoint* of L (or *dual map*) is the linear map $L^* : W^* \to V^*$ defined by setting $L^*(\xi) = \xi \circ L$ for all $\xi \in W^*$. The notation L^* is consistent with our notation "pullback of a real-valued function by a map": given a map $\xi : W \to \mathbf{R}$, and a map $L : V \to W$, we pull ξ back to a function on V simply by composing on the right with L. However, there is more going on.

Using composition, we *could* similarly pull back *any* function on W by *any* map $V \to W$, even if V and W were merely *sets*. But here they are vector spaces, and the map L is linear. Still, even knowing that $L : V \to W$ is a linear map between vector spaces, we could pull back *any* map $f : W \to \mathbf{R}$ (not necessarily linear) by L, and write $L^* f$ for the pulled-back map $f \circ L : V \to \mathbf{R}$. We could then view "$L^*$" as a map from $\{$all functions $W \to \mathbf{R}\}$ to $\{$all functions $V \to \mathbf{R}\}$, and this "grand" map L^* would even be linear.[2] But this is not customarily what the notation L^* means when L is linear.

Rather, in this context—where the essential ingredients are that V, W are vector spaces (rather than that they are manifolds, or arbitrary sets) and that L is linear—we restrict attention to pulling back *linear* maps $\xi : W \to \mathbf{R}$ to maps $\xi \circ L : V \to \mathbf{R}$. The resulting maps $\xi \circ L$ are themselves linear, and the map $W^* \to V^*$ given by $\xi \mapsto \xi \circ L$ is linear, so the "grand" pullback operation, by L, from $\{$*all* functions $W \to \mathbf{R}\}$ to $\{$*all* functions $V \to \mathbf{R}\}$, restricts to a linear map from $\{$*linear* functions $W \to \mathbf{R}\}$ to $\{$*linear* functions $V \to \mathbf{R}\}$. We reserve the notation L^* for this restricted pullback operation, a linear map $W^* \to V^*$.

Observe that using dual-pairing notation we could write the definition of L^* as:

$$\langle L^*\xi, v\rangle = \langle \xi, Lv\rangle \quad \forall \xi \in W^*,\ v \in V.$$

Hence, given manifolds M, N, a smooth map $F : M \to N$, and a point $p \in N$, the natural adjoint of the linear map $F_{*p} : T_pM \to T_{F(p)}N$ is the linear map $(F_{*p})^* : T^*_{F(p)}N \to T^*_pN$ defined by setting

$$\langle (F_{*p})^*\xi, v\rangle = \langle \xi, F_{*p}\, v\rangle \quad \forall \xi \in T^*_{F(p)}N,\ v \in T_pM. \tag{0.5}$$

(So, just as tangent vectors naturally push forward under general smooth maps, *co*vectors naturally pull back.)

The notation "$(F_{*p})^*$" is rarely used, since it is somewhat bewildering to look at: the upper-star is denoting a pullback *by the linear map* F_{*p}, which already contains a lower-star denoting a push-forward induced *by the map* F. Let us temporarily use the notation F^{*p} for $(F_{*p})^*$ (*temporarily* because the notation is unconventional.) Then the equation in (0.5) can alternatively be written as

$$(F^{*p}(\xi))(v) = \xi(F_{*p}\, v).$$

[2](1) Remember that for any nonempty set S and any vector space Z, the set $\mathrm{Maps}(S, Z)$ of all functions from S to Z has a natural vector-space structure, induced by pointwise operations ($(f+g)(p) = f(p) + g(p)$, etc.). Hence *linearity* of a map from one such map-space to another is well-defined. (2) Note that the *dimension* of $\mathrm{Maps}(S, \mathbf{R})$ is the *cardinality* of S. Thus $\mathrm{Maps}(V, \mathbf{R})$ has uncountable dimension if V is a nonzero vector space.

Thus, when $k = 1$, (0.4) says that $(F^*\omega)_p = F^{*p}(\omega_{F(p)})$. In other words, we pull back 1-forms by pulling back their values (covectors) pointwise, using the natural adjoints of the derivatives F_{*p}.

20. Let M be a manifold. Define $\Omega_0(M) = \bigoplus_{k \text{ even}} \Omega^k(M),\ \Omega_1(M) = \bigoplus_{k \text{ odd}} \Omega^k(M)$. Here we are regarding the *sub*scripts 0 and 1 as the two elements of the group $\mathbf{Z}_2$, indexing two vector spaces. This gives $\Omega^*(M)$ the structure of a $\mathbf{Z}_2$-graded vector space. Show that $\Omega^*(M)$, equipped with wedge-product, is a $\mathbf{Z}_2$-graded *algebra*.