Differential Geometry—MTG 6256—Fall 2025
Problem Set 1: Fun with Matrices

Due-date for required problems: Monday 9/22/25

General reminders for any handed-in work

Even when homework is well written, reading and grading it is very time-consuming and physically difficult for me. To keep this task from being more burdensome than it intrinsically needs to be:

1. Use *plain, white, unlined, printer paper with no holes.*

2. Make sure your work is *neat* and easy to read. It should either be typed (preferably in LaTeX) or written in pen or **dark** pencil, and there should be **no over-writing** (superimposing new writing on old, *with or without* erasure of the old writing first).

3. If you type your homework, use 12-point font. (LaTeX often defaults to 10-point font. To get 12-point font in, say, the "article" document class, the command I use is \documentclass[12pt]{article}.)

4. **Staple** your sheets together in the upper left-hand corner. Any other means of attachment makes more work for me. The staple should be close enough to the corner that when I turn pages, nothing that you've written is obscured. (If you have trouble getting the staple close enough to the corner to achieve this, you haven't left wide enough margins; see below.)

5. If you are writing on both sides of a sheet of paper, do not use paper/ink/pencil combinations for which the writing on one side of the paper shows on the other side (or darkens it).

6. Please use **wide** margins—at least 1.75"—on *all four edges* (left *and* right *and* top *and* bottom). LaTeX preamble commands that will accomplish this in the "article" document class are

   \setlength{\textwidth}{5 in}
   \setlength{\textheight}{7.3 in}
   \setlength{\oddsidemargin}{.75 in}
   \setlength{\topmargin}{0.2 in}

7. Make sure your sentences are **unambiguous**, as well as being correctly punctuated, grammatically correct, and complete.

Required problems (to be handed in; due-date 9/22/25): 2, 3, 4, 5 10ab. You should also read the footnote in problem 7. In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Optional problems: All the ones that are not required.

Items in blue are side-comments (sometimes intended as hints).

Below, $M_n(\mathbf{R})$ denotes the vector space of $n \times n$ real matrices, and $\mathrm{GL}(n, \mathbf{R})$ the subset of invertible matrices.

Several problems on this list were already done, or essentially done, in class and/or in the "Review of Advanced Calculus" notes; they are included as a guide and a reminder. For the same reason, some facts stated in class are stated again here. Problems 1–9 are meant to be done in the given order; in many cases the results of earlier problems are applicable to later problems. Problem 10 does not really use any of the other problems.

When asked to find the derivative of a map, express your answer by writing down a formula that gives all directional derivatives.

1. Let $U \in \mathbf{R}$ be open, W a finite-dimensional vector space, $f : U \to W$ differentiable at $p \in U$. Show that $Df|_p(1) = f'(p)$.

2. Let V, W_1, W_2 be finite-dimensional vector spaces of positive dimension, $U \subset V$ open, $p \in U$, and $g_i : U \to W_i$ differentiable at p for $i = 1, 2$. Define $f : U \to W_1 \oplus W_2$ by $f(q) = (g_1(q), g_2(q))$. Show that f is differentiable at p and compute $Df|_p(v)$ for arbitrary $v \in V$.

Note: one general approach to a problem of the form "show that a function F is differentiable at point q, and compute the derivative $DF|_q$" is to compute all the directional derivatives $(D_vF)_q$. If this expression is not linear in v, then F is not differentiable at q (and you were instructed to show something that was false). If $(D_vF)_q$ *is* linear in v, then the linear transformation T defined by $T(v) = (D_vF)_q$ is the *only candidate* for $DF|_q$. You can then try to show that F is differentiable at q either by plugging this T into the definition of "differentiable at q" and showing that the relevant limit is zero, or by showing that, for all fixed v, the map $\tilde{q} \mapsto (D_vF)_{\tilde{q}}$ is continuous in $\tilde{q}$ (in which case, automatically, F is not merely differentiable at q, but *continuously* differentiable at q). The former approach is the one to use in this problem; nothing in the hypotheses implies continuity of the directional derivatives.

3. Define $\mu : M_n(\mathbf{R}) \oplus M_n(\mathbf{R}) \to M_n(\mathbf{R})$ by $\mu(A, B) = AB$ (matrix multiplication). Show that μ is differentiable, and compute its derivative.

4. Let $g, h : M_n(\mathbf{R}) \to M_n(\mathbf{R})$ be differentiable and define $f(A) = g(A)h(A)$. Note that $f = \mu \circ j$, where μ is the map in problem 3 and $j : M_n(\mathbf{R}) \to M_n(\mathbf{R}) \oplus M_n(\mathbf{R})$ is defined by $j(A) = (g(A), h(A))$. Prove that f is differentiable, and (using directional

derivatives) express the derivative of f in terms of the derivatives of g and h.

If your answer is correct, then in the case $n = 1$, you should find with the aid of problem 1 that you've recovered the "product rule" from Calculus 1. Thus, the Calculus-1 product rule is a corollary of the (multivariable) Chain Rule.

5. Define $f : M_n(\mathbf{R}) \to M_n(\mathbf{R})$ by $f(A) = A^t A$, where A^t is the transpose of A. Show that f is differentiable and find its derivative.

Observe that $A^t A$ is always a *symmetric* matrix, so we can also define a map $f_1 : M_n(\mathbf{R}) \to \mathrm{Sym}_n(\mathbf{R})$ (the space of $n \times n$ symmetric matrices) by $f_1(A) = A^t A$. This fact will be used to prove something important in a later homework assignment. Note that for any $A, B \in M_n(\mathbf{R})$, we have $Df_1|_A(B) = (D_B f_1)|_A = (D_B f)|_A = Df|_A(B)$; the only difference between $Df_1|_A$ and $Df(A)$ is the codomain of the derivative ($\mathrm{Sym}_n(\mathbf{R})$ vs. $M_n(\mathbf{R})$).

6. Let $m \geq 1$ be an integer and let $f(A) = A^m$ for $A \in M_n(\mathbf{R})$. Show that f is differentiable and find its derivative.

7. We saw in class that $\mathrm{GL}(n, \mathbf{R})$ is an open subset of $M_n(\mathbf{R})$. Show that it is also dense. Hint for a quick proof: characteristic polynomial. (Look up the term if you've forgotten what it means, or never learned it.) Of course, there are many other proofs as well.

8. Define $\iota : \mathrm{GL}(n, \mathbf{R}) \to M_n(\mathbf{R})$ by $\iota(A) = A^{-1}$. In this exercise, you will show that ι is differentiable by a method different from the one sketched in class[1], without needing to show ahead of time that ι is continuous. (In class, I used the continuity of ι to deduce that directional derivatives were continuous, and said that continuity is not hard to show. That fact is true; we just don't need it in the approach below. Once differentiability is shown, we can deduce continuity of ι from the general fact that differentiability implies continuity.) The idea is the following:

- For a fixed A, we find a linear transformation T that is *the only possible candidate* for the derivative of ι at A.
- We plug this T into the quotient whose limit we take in the definition of "derivative", and show directly that the limit of the quotient is zero.

[1] Neither this method nor the power-series method in the "Review of Advanced Calculus" notes is the fastest way to show that ι is differentiable. The *fastest* way is simply to observe that there is an explicit formula for the inverse of an invertible matrix A, expressing the entries of A^{-1} as rational functions of the entries of A, where the denominator of each rational function is $\det(A)$ (see the last sentence of problem 10(c)). Thus, composing appropriately with an isomorphism between $M_n(\mathbf{R})$ and $\mathbf{R}^{n^2}$, ι becomes a map from an open set in $\mathbf{R}^{n^2}$ to $\mathbf{R}^{n^2}$, each of whose component-functions is a rational function with nonzero denominator. Each component-function is therefore not just differentiable, but C^∞, and therefore ι is actually a C^∞ map. This fact is important to know, independent of this homework problem. The purpose of this homework problem is intended to teach some other ideas related to the matrix-inversion map.

The same method can be used in many other examples. Usually the candidate T is found by computing directional derivatives, but in the case of ι there is a "cheaper" approach to finding the only candidate. This is part (a) below.

(a) Using the result of problem 4, show that <u>if</u> ι is differentiable, then $(D_A\iota)(B) = -A^{-1}BA^{-1}$ for all $A \in M_n(\mathbf{R}), B \in \mathrm{GL}(n, \mathbf{R})$. Note that for fixed A, the expression $-A^{-1}BA^{-1}$ is linear in B.

(b) Show that if $A \in M_n(\mathbf{R})$, then A is invertible if and only if A is *bounded below*, i.e. iff there exists $c > 0$ such that $\|Av\| \geq c\|v\|$ for all $v \in \mathbf{R}^n$.

(c) Using part (b) and the triangle inequality, show that for all $A \in \mathrm{GL}(n, \mathbf{R})$, there exists $\delta > 0$ such that if $\|B\| < \delta$ then $A + B$ is bounded below, hence is invertible. (This gives another proof that $\mathrm{GL}(n, \mathbf{R})$ is open in $M_n(\mathbf{R})$, of course.) Here and below, the norm used on $M_n(\mathbf{R})$ is the operator norm determined by the standard Euclidean norm on $\mathbf{R}^n$.

(d) Fix $A \in \mathrm{GL}(n, \mathbf{R})$ and define T to be the linear transformation found above in part (a), the map $B \mapsto -A^{-1}BA^{-1}$. Using just algebraic manipulation (e.g. the identity $X = C^{-1}(CXA)A^{-1}$ for $X \in M_n(\mathbf{R})$ and $A, C \in \mathrm{GL}(n, \mathbf{R})$) and the submultiplicativity of the operator norm, show that if $B \in M_n(\mathbf{R})$ and $A + B$ is invertible, then $\|\iota(A + B) - \iota(A) - T(B)\| \leq \|(A + B)^{-1}\| \; \|A^{-1}\|^2 \; \|B\|^2$.

(e) Show that if A and c are as in part (b), then $\|A^{-1}\| \leq \frac{1}{c}$.

Note: Since $1 = \|I\| = \|AA^{-1}\| \leq \|A\| \; \|A^{-1}\|$, we have the simple *lower* bound $\|A^{-1}\| \geq \|A\|^{-1}$. But there is no general *upper* bound on $\|A^{-1}\|$ in terms of $\|A\|$.

(f) Use part (e) to show that your work in part (c) actually gives you, for fixed A, a uniform-in-B upper bound on $\|(A + B)^{-1}\|$ if $\|B\|$ is sufficiently small.

(g) Now show that if $A \in \mathrm{GL}(n, \mathbf{R})$ then $\lim_{B \to 0} \frac{\|\iota(A+B) - \iota(A) - T(B)\|}{\|B\|} = 0$, hence that ι is differentiable at A, with derivative given by $D\iota|_A(B) = -A^{-1}BA^{-1}$.

9. Extend the result of problem 6 to negative integral exponents. (For $A \in \mathrm{GL}(n, \mathbf{R})$ and $m \geq 1$, A^{-m} is defined to be $(A^{-1})^m$.)

10. The determinant function $\det : M_n(\mathbf{R}) \to \mathbf{R}$ is a polynomial in n^2 variables, so it is certainly C^1 (in fact C^∞). There are several ways to compute its derivative. The steps below constitute a method that involves little computation but a bit of thought.

(a) Let $I \in M_n(\mathbf{R})$ be the identity matrix and let $B \in M_n(\mathbf{R})$. Compute $(D(\det))|_I(B)$, and express the answer as a simple invariant of the matrix B. (Note that since det is differentiable, its derivative at any point can be computed from directional derivatives: $D(\det)|_A(B) = (D_B \det)|_A$.)

(b) Let $A \in \mathrm{GL}(n, \mathbf{R}), B \in M_n(\mathbf{R})$. Compute $(D(\det))|_A(B)$. (Hint: use (a).) Re-express your result as a formula for the derivative (or directional derivatives) of the function $\log|\det|$.

(c) Use the density statement in problem 7 to extend the formula for $D_A(\det)$ from $A \in \mathrm{GL}(n, \mathbf{R})$ to $A \in M_n(\mathbf{R})$. The answer can be rewritten in terms of the "cofactor" matrix $\mathrm{cof}(A)$ that arises in computing the inverse of a matrix. (Recall that if A is invertible, then $A^{-1} = \frac{1}{\det(A)}\mathrm{cof}(A)$, or else the transpose of this, depending on your definition of $\mathrm{cof}(A)$.)