

Required problems (to be handed in): 1, 2abcf, 4d. In doing any of these problems or problem-parts, you may assume the results of all earlier problems or problem-parts (optional or required).

Optional problems: All the ones that are not required.

Below:

- “Atlas” always means an atlas whose charts all have the same dimension (in case the underlying topological space is not connected).
- Whenever we refer to an atlas on an already-fixed manifold, we mean an atlas within the (explicitly or implicitly) already-fixed maximal atlas.
- Any time a manifold M is constructed by giving an atlas on a set for which a topology is not stated in advance, the topology on M is taken to be the one induced by the atlas.
- I use the words “natural(ly)” and “canonical(ly)” without a formal, mathematically precise, universally applicable definition. Whenever one of these words comes up, it should be clear from context what it means in that context.
- “Map” always means “continuous map” (though sometimes “continuous” is inserted as a reminder, especially when I don’t want you to wonder whether I meant “smooth map”).
- “Smooth map of manifolds” means “smooth map from one manifold to another”.

1. Let M be a manifold, $U \subset M$ a nonempty open set. Show that an atlas on M naturally gives rise to an atlas on U of the same dimension, hence that U inherits a manifold structure.¹

2. *Real and complex projective spaces.* If V is a vector space over a field \mathbf{F} , the *projectivization* $P(V)$ is defined (as a set) to be the set of one-dimensional vector subspaces of V (“lines through the origin in V ”). Alternatively, $P(V) = (V \setminus \{0\}) / \sim$, where the equivalence relation \sim is defined by $v \sim w \iff v = tw$ for some $t \in \mathbf{F}$.

¹Once we define “submanifold”, an open subset of a manifold will be the most trivial example of a submanifold.

Let $n \geq 0$ and let $V = \mathbf{F}^{n+1}$, where \mathbf{F} is either \mathbf{R} or \mathbf{C} . Let $\pi : V \rightarrow P(V)$ be the quotient map, and denote $\pi(v)$ by $[v]$ whenever convenient. For $0 \leq j \leq n$ define

$$\begin{aligned}\tilde{U}_j &= \{(x^0, \dots, x^n) \in \mathbf{F}^{n+1} \mid x^j \neq 0\} \subset V, \\ U_j &= \pi(\tilde{U}_j) \subset P(V).\end{aligned}$$

Clearly $\{\tilde{U}_j\}_{j=0}^n$ covers $V \setminus \{0\}$, so $\{U_j\}_{j=0}^n$ covers $P(V)$. In view of the equivalence relation defining $P(V)$, the maps $\tilde{\phi}_j : \tilde{U}_j \rightarrow \mathbf{F}^n$ defined by

$$\tilde{\phi}_j(x^0, \dots, x^n) = \left(\frac{x^0}{x^j}, \dots, \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, \dots, \frac{x^n}{x^j} \right) \quad (1.1)$$

induce well-defined maps $\phi_j : U_j \rightarrow \mathbf{F}^n$,

$$\phi_j([v]) = \tilde{\phi}_j(v).$$

(In (1.1), it is understood that “ $\frac{x^0}{x^j}$ ” is omitted if $j = 0$ and that “ $\frac{x^n}{x^j}$ ” is omitted if $j = n$. Alternative notation for the right-hand side is

$$\left(\frac{x^0}{x^j}, \dots, \widehat{\frac{x^j}{x^j}}, \dots, \frac{x^n}{x^j} \right),$$

where the “hat” denotes omission of the term indexed by j .)

(a) Identify the sets $\phi_j(U_j)$ and $\phi_j(U_i \cap U_j)$ ($i \neq j$) explicitly, and compute the overlap-maps $\phi_i \circ \phi_j^{-1}$. (As always, in the overlap-map expression “ $\phi_i \circ \phi_j^{-1}$ ”, it is understood that “ ϕ_j ” is short-hand for “ $\phi_j|_{U_i \cap U_j}$ ”.)

(b) *Real projective space.* Show that $\{(U_i, \phi_i)\}_{i=0}^n$ is a smooth, n -dimensional atlas on $\mathbf{R}P^n := P(\mathbf{R}^{n+1})$. Hence $\mathbf{R}P^n$, with the corresponding maximal atlas, is an n -dimensional manifold.

Whenever anyone speaks of $\mathbf{R}P^n$ as a manifold, it's implicit that this is the smooth structure.

(c) *Complex projective space.* Any *real* isomorphism from the two-dimensional *real* vector space \mathbf{C} to \mathbf{R}^2 , such as $z \mapsto (\operatorname{Re}(z), \operatorname{Im}(z))$, induces a real isomorphism $\mathbf{C}^n \rightarrow \mathbf{R}^{2n}$ for $n \geq 1$. By composing the chart-maps ϕ_j with such an isomorphism, we obtain maps $U_j \rightarrow \mathbf{R}^{2n}$. To avoid notational clutter, in this problem we will abuse notation slightly, allowing “ ϕ_j ” to stand both for our previously-defined $U_j \rightarrow \mathbf{C}^n$ and for the corresponding map $U_j \rightarrow \mathbf{R}^{2n}$.

Show that $\{(U_i, \phi_i)\}_{i=0}^n$ is a smooth, $2n$ -dimensional atlas on $\mathbf{C}P^n := P(\mathbf{C}^{n+1})$. In the formula for $\phi_i \circ \phi_j^{-1}$, you may treat n -tuples of complex numbers as elements of \mathbf{R}^{2n} wherever necessary. Hence $\mathbf{C}P^n$, with the corresponding maximal atlas, is a $2n$ -dimensional manifold.

Whenever anyone speaks of $\mathbf{C}P^n$ as a manifold, it's implicit that this is the smooth structure.

Remark. There *is* such a thing as a complex manifold, and as you might conjecture, $\mathbf{C}P^n$ is a complex n -dimensional manifold. However, the concept is subtler than you might think, and for us, “manifold” will always mean “real manifold” unless otherwise specified.

(d) For $V = \mathbf{R}^{n+1}$ and $V = \mathbf{C}^{n+1}$, show that the topology on $P(V)$ induced by the atlases in parts (b) and (c) is the same as the quotient topology. (In case you need to review the meaning of *quotient topology*, it’s in the handout “Point-Set Topology: Glossary and Review” on the class home page.)

(e) Show that $\mathbf{C}P^1$ is diffeomorphic to S^2 by explicitly exhibiting a diffeomorphism $F : \mathbf{C}P^1 \rightarrow S^2$ that maps U_0 to $S^2 \setminus \{\text{north pole}\}$, and maps U_1 to $S^2 \setminus \{\text{south pole}\}$.

Remark. $\mathbf{C}P^1$ is also called the *Riemann sphere*. As a set, $\mathbf{C}P^1 = U_0 \coprod \{[(0, 1)]\}$ (“ \coprod ” means “disjoint union”). In the Riemann sphere, our set U_0 is implicitly identified with $\phi_0(U_0) = \mathbf{C}$, and the point $[(0, 1)]$ is regarded as “the point at infinity”. If you did part (a) correctly, you should find that both overlap maps are given by $z \mapsto \frac{1}{z}$, with domain $\mathbf{C} \setminus \{0\}$.

(f) Show that the quotient map (or *projection*) $\pi : V \setminus \{0\} \rightarrow P(V)$ is a smooth map of manifolds in the cases $V = \mathbf{R}^{n+1}$ and $V = \mathbf{C}^{n+1}$.

3. The *Grassmannian* or *Grassmann manifold* $G_k(\mathbf{R}^n)$ ($0 < k < n$) is a manifold whose underlying set is the set of k -dimensional subspaces of \mathbf{R}^n . (This is a generalization of real projective space; $G_1(\mathbf{R}^n) = P(\mathbf{R}^n) = \mathbf{R}P^{n-1}$. Notations for the Grassmannian vary in the literature: some people use the notation “ $G_k(\mathbf{R}^n)$ ” for the set of subspaces of \mathbf{R}^n of codimension k . The notations $G_{k,n}$ and $G_{n,k}$ are also used.)

To define a natural atlas on $G_k(\mathbf{R}^n)$, it is convenient to generalize our definition of “ N -dimensional chart” to allow charts with values in a fixed N -dimensional vector space V that we do not require to be (literally) \mathbf{R}^N . For any such V , we can fix an isomorphism $\iota : V \rightarrow \mathbf{R}^N$, and compose any V -valued chart-map with ι to obtain an \mathbf{R}^N -valued chart-map. If $f : (\text{open set in } V) \rightarrow (\text{open set in } V)$ is an overlap map for a pair of V -valued chart-maps, then the overlap map for the corresponding \mathbf{R}^N -valued chart-maps is $\iota \circ f \circ \iota^{-1}$, which is exactly as (continuously) differentiable as is f . In particular, an atlas of V -valued charts is smooth if and only if such a corresponding atlas of \mathbf{R}^N -valued charts is smooth. Thus, allowing V -valued charts does not change the collection of objects we call smooth N -dimensional manifolds.

With this in mind, the “standard” smooth atlas on $G_k(\mathbf{R}^n)$ is constructed as follows. Endow \mathbf{R}^n with the standard inner product. Observe that given any k -plane X through the origin, any sufficiently close k -plane Y through the origin is the “orthogonal graph” of a unique linear map $T : X \rightarrow X^\perp$, where X^\perp is the orthogonal complement of X . (“Sufficiently close” means that $Y \cap X^\perp = \{0\}$. “Orthogonal graph of T ” means $\{v + T(v) \mid v \in X\}$. Since $\mathbf{R}^n = X \oplus X^\perp$, there is a natural bijection between \mathbf{R}^n and $X \times X^\perp$. Composing appropriately with this bijection identifies the orthogonal graph of T with the “true” graph of T .) For each k -element

subset $I = \{i_1, i_2, \dots, i_k\}$ of $\{1, 2, \dots, n\}$, let X_I be the subspace consisting of all $x \in \mathbf{R}^n$ all of whose coordinates other than those in positions i_1, \dots, i_k vanish. Let $V_I \subset \mathbf{R}^n$ be the (set-theoretic) complement of X_I^\perp in \mathbf{R}^n .

(a) Let $\mathcal{I} = \{(i_1, i_2, \dots, i_k) \in \mathbf{N}^k : 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$. Show that $\{V_I\}_{I \in \mathcal{I}}$ is an open cover of $\mathbf{R}^n \setminus \{0\}$ and determines a cover $\{U_I\}$ of $G_k(\mathbf{R}^n)$, that, for $k = 1$, reduces to the cover used in problem 2b (modulo replacing \mathbf{R}^{n+1} with \mathbf{R}^n).

(b) Show that there is a natural bijection from U_I to $\text{Hom}(X_I, X_I^\perp)$, a space naturally isomorphic to $M_{(n-k) \times k}(\mathbf{R})$ (the space of $(n-k) \times k$ real matrices). Hence there is a natural bijection $\phi_I : U_I \rightarrow M_{(n-k) \times k}(\mathbf{R})$.

(c) Show that the overlap maps $\phi_J \circ \phi_I^{-1}$ are smooth (this requires quite a bit more work for general k than did the $k = 1$ case in problem 2), and hence that $G_k(\mathbf{R}^n)$ is a smooth manifold of dimension $k(n-k)$.

4. *Covering spaces.* A *covering space* of topological space X is a pair (\tilde{X}, π) , where \tilde{X} is a topological space and $\pi : \tilde{X} \rightarrow X$ is a continuous surjective map with the following property: for each $p \in X$, there is an open neighborhood U of p that is *evenly covered*, meaning that $\pi^{-1}(U)$ is a union of disjoint open sets in \tilde{X} , each of which is mapped homeomorphically by π to U . (Surjectivity is automatic if “union” is replaced by “non-empty union”.) The map π is called the *projection* or the *covering map*.

Below, assume that M, N are manifolds and that $(\tilde{M}, \pi), (\tilde{N}, \pi')$ are covering spaces of M, N respectively.

(a) Let $m = \dim(M)$. Show that an atlas on M gives rise to an m -dimensional atlas on \tilde{M} , hence that \tilde{M} naturally inherits the structure of a smooth m -dimensional manifold. (For this reason we usually refer to \tilde{M} or (\tilde{M}, π) as a covering *manifold* of M , rather than just a covering *space*.)

(b) **Definition.** Let X, Y be manifolds. A map $F : X \rightarrow Y$ is a *local diffeomorphism* if F is an open map and every $p \in X$ has an open neighborhood U with the property that $F|_U : U \rightarrow F(U)$ is a diffeomorphism. Here, $F(U)$ is an open set since the map F is open, and thus $F(U)$ naturally carries a manifold structure by problem 1. (In part (f) of this problem, we will see that this definition is equivalent to a briefer, more standard, but less self-explanatory one.)

Show that the natural smooth structure (equivalently, maximal atlas) on \tilde{M} in part (a) is the unique smooth structure for which π is a local diffeomorphism.

(c) Suppose that \tilde{M} is a manifold of dimension m . Assume that for any two open sets $\tilde{U}_1, \tilde{U}_2 \subset \tilde{M}$ for which $\pi|_{\tilde{U}_i}$ is injective and for which $\pi(\tilde{U}_1) = \pi(\tilde{U}_2)$, the map $(\pi|_{\tilde{U}_2})^{-1} \circ \pi|_{\tilde{U}_1}$ is smooth. (Hence all such maps are diffeomorphisms.) Show that M naturally inherits the structure of a smooth m -dimensional manifold.

(d) Let $F : M \rightarrow N$ be a (continuous) map, and let \tilde{F} be a (continuous) map

either from \widetilde{M} to N , from M to \widetilde{N} , or from \widetilde{M} to \widetilde{N} . If the corresponding diagram in Figure 1 commutes, we call \tilde{F} a *lift* of F . (Figure 1 is somewhere near this paragraph, wherever LaTeX felt like putting it.) Show that for lifts of all three types, cases, if \tilde{F} is a lift of F , then \tilde{F} is smooth if and only if F is smooth.

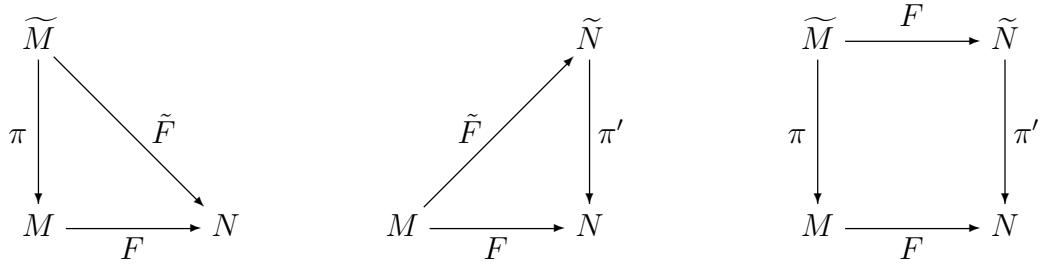


Figure 1: Diagrams for problem 4

Note: Given only $F : M \rightarrow N$, a unique lift $\tilde{F} : \widetilde{M} \rightarrow N$ always exists, namely $\tilde{F} := F \circ \pi : M \rightarrow N$. The other two types of lifts do not always exist, and when they exist, they may not be unique. Given only \tilde{F} (of any of the three types indicated with this notation above), we say that \tilde{F} *descends* to a map $M \rightarrow N$ if there exists $F : M \rightarrow N$ of which \tilde{F} is a lift. A map $\tilde{F} : M \rightarrow \widetilde{N}$ always descends uniquely to the map $F := \pi' \circ \tilde{F}$; in the other two cases, \tilde{F} does not always descend, but when it does descend, the map to which it descends is unique.