Differential Geometry—MTG 6256—Fall 2025
Problem Set 3
Due-date: Wednesday 11/19/25

Required problems (to be handed in): 1, 4a, 6a, 9, 10, 12abd. In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Optional problems: All the ones that are not required.

Throughout this assignment, "manifold" means "Hausdorff manifold".

Several problems on this assignment were assigned at the blackboard over several weeks, and/or were partially done in class. We've used a number of the results of these in class already.

1. Let $\mathrm{Sym}_n(\mathbf{R}) \subset M_n(\mathbf{R}) \cong \mathbf{R}^{n^2}$ be the vector subspace consisting of $n \times n$ symmetric matrices (those A for which $A^t = A$). Define $F : M_n(\mathbf{R}) \to \mathrm{Sym}_n(\mathbf{R})$ by $F(A) = A^t A$. Let $I \in M_n(\mathbf{R})$ be the identity; note that $F^{-1}(I) = \{A \in M_n(\mathbf{R}) \mid A^t A = I\}$, which is also known as the *orthogonal group* $O(n)$. Show that I is a regular value of F, and hence that $O(n)$ is a submanifold of $M_n(\mathbf{R})$. What is the dimension of $O(n)$?

Note: $O(n)$ is not connected; it has two connected components, the set $SO(n)$ of orthogonal matrices of determinant 1, and the set of orthogonal matrices of determinant -1. This example shows that non-connected manifolds can arise naturally in important examples.

2. Let M be a manifold of dimension m, and N a nonempty subset of M. (The additional hypotheses in each part below are independent of each other; they do not continue from one part to the next.)

(a) Let M' be another manifold and let $F : M \to M'$ be a diffeomorphism. Show that N is a submanifold of M if and only if $F(N)$ is a submanifold of M'. (This can be done directly, or deduced as a consequence of problem 7 in this assignment.)

(b) Let (U, φ) be a chart of M. Show that, viewed as a map from U to $\varphi(U) \subset \mathbf{R}^n$, the map φ is a diffeomorphism.

Remark: Together, parts (a) and (b) imply the following: given any chart (U, φ) of M (not necessarily adapted to N), the set $N \bigcap U$ is a submanifold of U (hence of M) if and only if $\varphi(N \bigcap U)$ is a submanifold of $\varphi(U)$ (hence of $\mathbf{R}^m$).

(c) Show that if for every $p \in N$ there exists an M-open neighborhood U of p such that $N \bigcap U$ is a submanifold of U, then N is a submanifold of M.

3. Let M, N be manifolds, let $F : M \to N$ be a smooth map, and let $p \in M$. Show

that F_{*p} is an isomorphism if and only if there exist an open neighborhoods U of p, and V of $F(p)$, such that $F|_U$ is a diffeomorphism from U to V. (Thus we can re-define "local diffeomorphism $F : M \to N$" to be a smooth map for which F_{*p} is an isomorphism for all $p \in M$.)

4. Let M, N be manifolds.

(a) Let $Z \subset N$ be a submanifold of N, and let $F : M \to N$ be a not-necessarily-smooth map whose image lies in Z. Show that F is smooth as a map from M to N if and only if F is smooth as a map from M to Z. (Here, "smooth as a map from M to Z" means that the map $\hat{F} : M \to Z$ obtained by replacing the codomain of F with Z, is smooth.) Keep in mind that there is no such thing as "proof by lack of imagination" ("this is true because why wouldn't it be true?", "this is true because I don't see how it could be false," etc.). Your argument will need to *use* the definition of "submanifold" and "smooth map from one manifold to another" (applied both to F and $\hat{F}$), not just assert that the result follows from these definitions.

(b) Now let $Z \subset M$ be a submanifold of M. Show that if $F : M \to N$ is smooth, then $F|_Z : M \to N$ is smooth. (Again, you need to use the definition of "smooth map from one manifold to another"; the definition of "smooth map from Z to N" is *not* "restriction of a smooth map from M to N.")

5. Let M, N be manifolds, with M compact and N connected. Show that if $F : M \to N$ is a submersion, then F is surjective.

6. (a) Let M, N be manifolds *of equal dimension*, with M compact and N connected. Prove that if M can be embedded in N, then M and N are diffeomorphic. (Thus, for example, the sphere S^2 cannot be embedded in the torus, or vice-versa.)

(b) Show that the assertion in part (a) would be false if the assumption "M is compact" were removed.

(c) Show that the assertion in part (a) would be false if the assumption "N is connected" were removed.

7. Let M be a manifold, $N \subset M$ a submanifold, and $j : N \to M$ the inclusion map. Show that j is an embedding.

8. Let M, N be manifolds and let $F : N \to M$ be an embedding. Show that $F(N)$ is a submanifold of M.

9. Let $F : M \to N$ be a smooth map of manifolds, let $q \in \text{image}(F)$, and assume that q is a regular value of F. Then, by the Regular Value Theorem, $Z := F^{-1}(q)$ (the *fiber of F over q*) is a submanifold of M. Let $j : Z \to M$ be the inclusion map, and let $p \in Z$. Show that

$$j_{*p}(T_pZ) = \ker(F_{*p}).$$

(In words: the tangent space at p to the fiber containing p is the kernel of derivative of F at p.)

10. Let M and N be manifolds of dimensions m and n respectively. For $p \in M$, $q \in N$, how is $T_{(p,q)}(M \times N)$ related to T_pM and T_qN?

11. Let $G_k(\mathbf{R}^n)$ be the Grassmannian of k-dimensional subspaces of $\mathbf{R}^n$, $0 \le k \le n$. (See Problem Set 2.) Define $F : G_k(\mathbf{R}^n) \to G_{n-k}(\mathbf{R}^n)$ by $X \mapsto X^\perp$. Show that F is a diffeomorphism. (Thus, for example, $G_2(\mathbf{R}^3)$ is diffeomorphic to $\mathbf{R}P^3$).

12. **Transversality.** This multi-part problem deals with a very important concept and tool in differential topology; for example part (e) is essential to the differential-topological definition and interpretation of the *degree* of a smooth map from one compact n-dimensional manifold to another. (We do not give that definition in this problem.)

Notation: Given two vector subspaces U, V of a vector space W, we define their *sum* $U + V$ to be the subspace $\{u + v \mid u \in U, v \in V\}$ (also called span$\{U, V\}$).[1]

Two submanifolds M and Z of a manifold N are said to intersect *transversely* at a point $z \in N$ if $T_zM + T_zZ = T_zN$ (more precisely, if $\iota_{*z}(T_zM) + j_{*z}(T_zZ) = T_zN$, where ι, j are the inclusion maps of M, Z, respectively, into N). If this condition is met at all points of $M \bigcap Z$ we say simply that M and Z *intersect transversely*, or have *transverse intersection*, or that *the intersection is transverse*, and write $M \pitchfork Z$.

More generally, given manifolds M, N and a submanifold $Z \subset N$, a map $F : M \to N$ is said to be *transverse to* Z if for all $(p, z) \in M \times Z$ with $F(p) = z$, we have $F_{*p}(T_pM) + T_zZ = T_zN$. Short-hand notation for "F is transverse to Z" is "$F \pitchfork Z$". The special case in which F is the inclusion map of a submanifold of N then reduces to the previous paragraph's definition of transverse intersection of two submanifolds of N.

Transversality comes into play when we ask the question "Is the intersection of two submanifolds a submanifold?" The answer is no in general, but yes if the intersection is transverse. Transversality is a *sufficient*, but not necessary, condition for the intersection to be a submanifold. Some examples with $N = \mathbf{R}^3$, with standard coordinates x, y, z: (i) the submanifolds $Z = xy$-plane, $M = yz$-plane, intersect transversely; (ii) the submanifolds $Z = xy$-plane, $M = z$-axis, intersect transversely; (iii) the submanifolds $Z = x$-axis, $M = y$-axis, do not intersect transversely ; (iv) the

[1] Note that U and V are allowed to have nontrivial intersection. When the intersection is trivial, i.e. $U \bigcap V = \{0\}$, we say that W is the *direct* sum of U and V, and (sometimes) write $W = U \oplus V$. However, we also use the symbol "$\oplus$" for the direct sum of two arbitrary vector spaces that aren't given to us as subspaces of a third.

submanifolds $Z = xy$-plane, $M = \{\text{graph of } z = x^2 - y^2\}$, do not intersect transversely (because of what happens at the origin).

(a) Let $N = \mathbf{R}^n$, $0 \leq k \leq n$, and view N as $\mathbf{R}^k \times \mathbf{R}^{n-k}$. (For the cases $k = 0$ and $k = n$, the convention is $\mathbf{R}^0 = \{0\}$ and we make the obvious identifications of $\{0\} \times \mathbf{R}^n$ and $\mathbf{R}^n \times \{0\}$ with $\mathbf{R}^n$.) Let Z be the k-dimensional submanifold $\mathbf{R}^k \times \{0 \in \mathbf{R}^{n-k}\}$. Prove that if M is a manifold and $F : M \to N$ is transverse to Z, then $F^{-1}(Z)$ is a submanifold of M. (*Hint*: Consider the map $G = \pi \circ F : M \to \mathbf{R}^{n-k}$, where $\pi : \mathbf{R}^k \times \mathbf{R}^{n-k} \to \mathbf{R}^{n-k}$ is projection onto the second factor.)

("Pre-b") Let X and Y be vector spaces, $W \subset Y$ a (vector) subspace, and let $L : X \to Y$ be a linear map. (i) Show that L induces a well-defined, injective linear map $\bar{L} : X/L^{-1}(W) \to Y/W$. (If you are not familiar with *quotient* [vector] *spaces*, or have forgotten what they are, look them up.) (ii) Show that $\bar{L}$ is surjective if and only if $L(X) + W = Y$. (iii) Now observe, for later use, that as a consequence of (i) and (ii), if $L(X) + Y = W$ then $X/L^{-1}(W) \cong Y/W$, implying that the codimension of $L^{-1}(W)$ in X equals the codimension of W in Y. When X and Y are finite-dimensional, this equality of codimensions yields an equation relating $\dim(X), \dim(Y), \dim(W)$, and $\dim(L^{-1}(W)$.

(b) Use the result of part (a) to prove that if M, N are arbitrary manifolds and $F : M \to N$ is transverse to a submanifold $Z \subset N$, then $F^{-1}(Z)$ is a submanifold of M. (Note that the case $Z = \{\text{point}\}$ is the Regular Value Theorem, so the theorem you're asked to prove here may be considered a generalization of the Regular Value Theorem.) What are the dimension and codimension of $F^{-1}(Z)$?

(c) Part (b), applied to the case in which F is the inclusion map of a submanifold $M \subset N$, shows that if $M \pitchfork Z$ and $M \bigcap Z \neq \emptyset$, then $M \bigcap Z$ is a submanifold of M. For $p \in M \bigcap Z$, express $T_p(M \bigcap Z)$ in terms of T_pM and T_pZ, and express $\dim(M \cap Z) = \dim(T_p(M \bigcap Z))$ in terms of $\dim(M), \dim(Z)$, and $\dim(N)$.

(d) In the setting of part (c), $M \bigcap Z$ is also a submanifold of Z, by symmetry. It is easy to show that a submanifold of a submanifold of N is, itself, a submanifold of N, so:

- M is a submanifold of N, of a certain codimension;
- Z is a submanifold of N, of a certain codimension;
- $M \bigcap Z$ is a submanifold of M, of a certain codimension;
- $M \bigcap Z$ is a submanifold of Z, of a certain codimension; and
- $M \bigcap Z$ is a submanifold of N, of a certain codimension.

Express the last three codimensions on this list in terms of the first two. To understand what these relations are saying, after you figure out the formulas, write them out without choosing letters to represent dimensions or codimensions; i.e. using the terms "codimension of M in N", "codimension of $M \bigcap Z$ in M", etc.

(e) Independent of the earlier parts of this problem, what is a necessary and sufficient condition that a subset S of a given manifold be a zero-dimensional submanifold? (The condition should involve nothing more than point-set topology.) Apply this condition when M, Z are transversely-intersecting submanifolds of N of complementary dimensions ($\dim(M) + \dim(Z) = \dim(N)$). What do you conclude about $M \bigcap Z$ in this case? If both M and Z are compact, what stronger conclusion can you reach?