Differential Geometry—MTG 6256—Fall 2025
Problem Set 4
Due-date: Wednesday 12/3/25

Required problems (to be handed in): 2bc, 3bc, 4bdef. In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Optional problems: All the ones that are not required.

Throughout this assignment, "manifold" means "paracompact, Hausdorff manifold".

Some problems in this assignment were assigned at the blackboard and/or were partially done in class.

1. Recall that a topological space X is *arcwise connected* (or *path-connected*) if for all $p, q \in X$, there exists a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$. It is easily shown that every arcwise connected space is connected (a separation of X would lead to a separation of $[0,1]$), but there are connected spaces that are not arcwise connected (the famous example being the "topologist's sine curve").

Show that a manifold M is connected if and only if M is arcwise connected. (You may assume the "arcwise connected $\implies$ connected" half of this iff; you need only show the "connected implies arcwise connected" half.)

Note: This problem was inserted here because its result can be used to simplify arguments in some later problem-parts involving connectedness. However, most of those problem-parts can be done without any reliance on arcwise-connectedness.

2. Let $n \geq 1$, let M and N be oriented n-dimensional manifolds, and let $F : N \to M$ be a smooth map. Recall that at any point of M or N, a basis of the tangent space is called either *positively oriented* or *negatively oriented*, according to whether basis is or is not in the orientation class defined that manifold's given orientation.

(a) Let $p \in N$ and suppose that the derivative $F_{*p} : T_pN \to T_{F(p)}M$ is an isomorphism. Show that if F_{*p} carries *some* positively oriented basis of T_pN to a positively oriented basis of $T_{F(p)}M$, then F_{*p} carries *every* positively oriented basis of T_pN to a positively oriented basis of $T_{F(p)}M$. Similarly, show that if F_{*p} carries *some* positively oriented basis of T_pN to a negatively oriented basis of $T_{F(p)}M$, then F_{*p} does that to *every* positively oriented basis of T_pN.

Part (a) shows that the following definition is unambiguous.

Definition. For a given $p \in N$, we say that F is *orientation-preserving* *at p* (respectively, *orientation-reversing at p*) if F_{*p} carries positively oriented bases of T_pN to positively (respectively, negatively) oriented bases of $T_{F(p)}M$. We say that F is *orientation-preserving* (respectively, *orientation-reversing*) if F is orientation-preserving at every $p \in M$ (respectively, orientation-reversing at every $p \in M$).

Note that for F to be either orientation-preserving or orientation-reversing at a point p, the map F_{*p} must be an isomorphism. Hence the only maps $N \to M$ that can *possibly* be orientation-preserving or orientiation-reversing (globally) are local diffeomorphisms.

(b) For any $p \in N$ or $q \in M$, recall that the given manifold-orientations also define what we mean by *positive* and *negative* elements of the 1-dimensional vector space $\bigwedge^n T_p^*N$ or $\bigwedge^n T_q^*M$. Show that F is orientation-preserving at $p \in N$ (respectively, orientation-reversing at $p \in N$) if and only if the pullback map $F^* : \bigwedge^n T_{F(p)}^*M \to \bigwedge^n T_p^*N$ carries some, and hence any, positive element of $\bigwedge^n T_{F(p)}^*M$ to a positive (respectively, negative) element of $\bigwedge^n T_p^*N$.

(c) Assume that N is connected and that $F : N \to M$ is a diffeomorphism. (i) Show that F is either orientation-preserving or orientation-reversing. (ii) Let $\omega \in \Omega_c^n(M)$ (the space of n-forms of compact support). Show that $F^*\omega$ has compact support (ensuring that $\int_N F^*\omega$ is defined), and prove the following:

$$\int_N F^*\omega = \pm \int_M \omega,$$

with the plus sign if F preserves orientation, and the minus sign if F reverses orientation. (This fact is called *invariance of the integral under diffeomorphism*.)

3. Let M and $\widetilde{M}$ be manifolds of equal dimension, and assume that $F : \widetilde{M} \to M$ is a submersion. Note that, for dimensional reasons (and using a previous homework problem), "F is a submersion" is equivalent to "F is a local diffeomorphism."

(a) Show that an orientation of M (if one exists) induces, via F, an orientation on $\widetilde{M}$. (Hence if M is orientable, so is $\widetilde{M}$.)

For the remaining parts of this problem, assume that $\widetilde{M}$ is compact and that M is connected. Since F is already assumed to be a submersion, a previous homework problem shows that F is surjective. Hence M $(= F(\widetilde{M}))$ is compact as well.

(b) Prove that F is a smooth covering map; i.e. that for all $p \in M$ there exists an open neighborhood U of p such that $F^{-1}(U)$ is a disjoint union of sets $\tilde{U}_i$ for which $F|_{\tilde{U}_i} : \tilde{U}_i \to U$ is a diffeomorphism. (Here i runs over some index set $\mathcal{I}(p)$, possibly depending on p.)

(c) Prove that for all $p \in M$, the set $F^{-1}(p)$ is finite. (Recall that "$F^{-1}(p)$" is common, though imprecise, notation for $F^{-1}(\{p\})$.)

(d) Prove that the cardinality of the finite set $F^{-1}(p)$ is independent of p. This finite common value—the number of points in the pre-image of any $p \in M$—is called the *degree* of F as a covering map.[1] (More generally, we may use this definition of degree of a covering map F any time the cardinality of $F^{-1}(p)$ is independent of p, whether or not $\widetilde{M}$ is compact or M is connected.) Below, we use the notation $\deg F$ for this degree.

(e) Assume that M is oriented, and give $\widetilde{M}$ the induced orientation. Show that for all $\omega \in \Omega^n(M)$,

$$\int_{\widetilde{M}} F^*\omega = (\deg F) \int_M \omega.$$

(The compactness of $\widetilde{M}$ and M ensures that both integrals are defined.)

4. Let M be an n-dimensional manifold, $n \geq 1$. We can construct a manifold called the *orientation double-cover* $\widetilde{M}$ of M as follows. For each $p \in M$ let $\mathrm{Orn}(p)$ denote the set of orientations of T_pM, a two-element set. Given $\sigma \in \mathrm{Orn}(p)$, we let $-\sigma$ denote the other orientation. As a set, let $\widetilde{M} = \bigcup_{p \in M} \mathrm{Orn}(p)$. There is a natural two-to-one map $\pi : \widetilde{M} \to M$ carrying both elements of $\mathrm{Orn}(p)$ to p. We give $\widetilde{M}$ the topology induced by this map π (i.e. a set $\tilde{U} \subset \widetilde{M}$ is declared to be open if and only if $\pi(\tilde{U})$ is open).

It can be shown that every manifold has as an atlas $\{(U_\alpha, \varphi_\alpha)\}$ for which all the sets U_α and nonempty intersections $U_\alpha \bigcap U_\beta$ are connected.[2] Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be such an atlas for M. Then, for each $\alpha \in A$, the set $\pi^{-1}(U_\alpha)$ has two connected components, which are distinguished from each other as follows. For $p \in U_\alpha$ let $\sigma_\alpha(p)$ be the orientation of T_pM pulled back by the map $\varphi_\alpha : U_\alpha \to \mathbf{R}^n$, where $\mathbf{R}^n$ is given its standard orientation. Each $\tilde{p} \in \pi^{-1}(U_\alpha)$ is, by definition, an orientation of $T_{\pi(p)}M$; hence $\tilde{p} = \pm\sigma_\alpha(\pi(\tilde{p}))$ (where "$+\sigma$" means σ). The sign in this formula is constant on each connected component of $\pi^{-1}(U_\alpha)$ (why?). We define $\tilde{U}_{\alpha,+}$ to be the component on which $\tilde{p} = \sigma_\alpha(\pi(\tilde{p}))$, and $\tilde{U}_{\alpha,-}$ to be the component on which $\tilde{p} = -\sigma_\alpha(\pi(\tilde{p}))$. We define corresponding chart-maps $\tilde{\varphi}_{\alpha,\pm} : \tilde{U}_{\alpha,\pm} \to \mathbf{R}^n$ as follows. Let $r : \mathbf{R}^n \to \mathbf{R}^n$ be the reflection $(x^1, x^2, \ldots x^n) \mapsto (-x^1, x^2, \ldots, x^n)$. Then we define $\tilde{\varphi}_{\alpha,+} = \varphi_\alpha \circ \pi$,

[1] In this coarse usage of the word "degree" for covering maps, the degree is always positive. For more general maps between compact, oriented manifolds of equal dimension, there is a notion of degree in which the degree can be positive, negative, or zero. For example, if $\widetilde{M} = M = S^1 =$ unit circle in $\mathbf{C}$, for $0 \neq n \in \mathbf{Z}$ the degree of the map $z \mapsto z^n$, as defined in this problem, is $|n|$. But for these maps it makes sense to refine the definition of degree, and even include the case $n = 0$, declaring the degree of $z \to z^n$ to be n whether this integer is positive, negative, or zero. This refined degree then classifies homotopy classes of maps $S^1 \to S^1$; every continuous map is homotopic to $z \mapsto z^n$ for a unique $n \in \mathbf{Z}$.

[2] It takes some non-trivial work to show this. Just assume it's true for now.

$\tilde{\varphi}_{\alpha,-} = r \circ \varphi_\alpha \circ \pi$.

(a) Let $\tilde{A} = A \times \{+,-\}$, an index set for the pairs $(\tilde{U}_{\alpha,\pm}, \tilde{\varphi}_{\alpha,\pm})$ constructed above. Show that $\{\tilde{U}_{\tilde{\alpha}}, \tilde{\varphi}_{\tilde{\alpha}}\}_{\tilde{\alpha}\in\tilde{A}}$ is an atlas for $\widetilde{M}$, hence that $\widetilde{M}$ is a manifold. (You may assume that paracompactness and Hausdorffness of M imply that $\widetilde{M}$ also has these properties. This is not hard to show, but your time would be better spent on other problems in this assignment.)

(b) Show that the atlas $\{\tilde{U}_{\tilde{\alpha}}, \tilde{\varphi}_{\tilde{\alpha}}\}_{\tilde{\alpha}\in\tilde{A}}$ is oriented (whether or not M is orientable!). Hence $\widetilde{M}$ is orientable; even better, the construction above gives it a *canonical orientation*, the one induced by this atlas. (It can be shown that this orientation is independent of the atlas of M that we started with, but I'm not asking you to show that.)

(c) Show that $\pi : \widetilde{M} \to M$ is a (smooth), degree-two covering map.

Discussion to set up part (d). From the definition of "covering map", it is easily shown that $\widetilde{M}$ has the following "path-lifting property": given any continuous map $\gamma : [0,1] \to M$, and any $\tilde{p} \in \pi^{-1}(\gamma(0))$, there exists a unique continuous map $\tilde{\gamma} : [0,1] \to \widetilde{M}$ with $\tilde{\gamma}(0) = \tilde{p}$. (You may assume this, but you should be able to prove it on your own, using nothing about manifolds other than that they are topological spaces. The degree of the cover, or even whether the cover *has* finite degree, is also irrelevant. The same argument works just as easily for any covering space of any topological space.) For general covering spaces, such a curve $\tilde{\gamma}$ is called a *lift* of γ; in the context of the orientation double-cover $\widetilde{M} \xrightarrow{\pi} M$ we may call such a lift an "orientation of M along γ".

(d) Assume M is connected. Show that M is orientable if and only if $\widetilde{M}$ is *not* connected. (Thus, if we start with a non-orientable, connected M, we obtain a counterexample to the [false] converse of the parenthetic conclusion of problem 3(a).) For the case in which M is orientable, show that $\widetilde{M}$ is diffeomorphic to $M \times \mathbf{Z}_2$, the disjoint union of two "copies" of M.

(e) Since every point in $\widetilde{M}$ is an orientation of a vector space, there is a natural map $\tau : \widetilde{M} \to \widetilde{M}$ defined by $\tau(\sigma) = -\sigma$. (This map is called an *involution*—a term you may recall from group theory—because $\tau \circ \tau$ is the identity map but τ itself is not.) Show that τ is an orientation-reversing map (where $\widetilde{M}$ is given the canonical orientation defined in part (b)).

(f) Since $\widetilde{M}$ is oriented, we may integrate any compactly supported n-form over $\widetilde{M}$. Show that if $\omega \in \Omega^n(M)$ is compactly supported, then so is $\pi^*\omega$, and

$$\int_{\widetilde{M}} \pi^*\omega = 0. \tag{0.1}$$

Hint for showing (0.1) quickly and elegantly: part (e).

5. Let $\widetilde{M}$ be a manifold and suppose that $F : \widetilde{M} \to \widetilde{M}$ is a smooth involution with no fixed-points. (Thus $F \circ F = \mathrm{id}_M$, and for every $p \in M$, $F(p) \neq p$.) Let $\sim$ be the equivalence relation on $\widetilde{M}$ generated by declaring $p \sim F(p)$. (Thus, the equivalence class of p is the set $\{p, F(p)\}$.) Let $M = \widetilde{M}/\sim$, with the quotient topology (if you're unsure what this topology is, see p. 5 of the topology glossary linked to the class home page).

(a) Show that for every manifold N, every smooth involution $\tau : N \to N$ is a diffeomorphism (whether or not τ has any fixed points).

(b) Show that each $p \in \widetilde{M}$ has an open neighborhood U such that $U \cap F(U) = \emptyset$.

(c) Show that the quotient-construction defining M determines, canonically, a smooth structure on M.

(Idea: Show that $\widetilde{M}$ has an atlas $\tilde{\mathcal{A}}$ such that the domain U of every chart in $\tilde{\mathcal{A}}$ satisfies $U \cap F(U) = \emptyset$. Use such an atlas to construct an atlas $\mathcal{A}$ of M. Show that if we apply this construction to any two atlases of $\widetilde{M}$ [within the given maximal atlas of $\widetilde{M}$] that have this "$U \cap F(U) = \emptyset$" property, the atlases of M we obtain are compatible, and hence determine the same smooth structure on M. The last step is necessary since atlases $\tilde{\mathcal{A}}$ of the type above are not unique.)

For the remainder of this problem, we regard M (the quotient space defined in the problem setup) as a manifold with the above natural smooth structure.

(d) Assume that $\widetilde{M}$ is orientable.

(i) Show that if F is orientation-preserving, then M is orientable.

(ii) Show that if F is orientation-reversing *and $\widetilde{M}$ is connected*, then M is not orientable. (Note that to show that M is not orientable, it's not sufficient to produce a non-oriented atlas! Every manifold, whether or not orientable, has non-oriented atlases.)

Hint: Choose any $\tilde{p} \in \widetilde{M}$. If $\widetilde{M}$ is connected, problem 2 assures us that there is a path in $\widetilde{M}$ from $\tilde{p}$ to $F(\tilde{p})$. Consider the image of this curve under the projection $\widetilde{M} \to M$; note that the curve in $\widetilde{M}$ is a lift of the curve in M. Show that assuming M is oriented leads to a contradiction.

6. Let $n \geq 1$. For each $p \in \mathbf{R}^{n+1}$, let ι_p denote the canonical isomorphism $T_p\mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$. Recall that the *standard orientation of* $\mathbf{R}^{n+1}$ *(as a vector space)* is

the orientation-class of the standard basis of $\mathbf{R}^{n+1}$. Regarding $\mathbf{R}^{n+1}$ as a manifold with a one-chart atlas $\{(\mathbf{R}^{n+1}, \mathrm{id})\}$, we obtain the *standard orientation of* $\mathbf{R}^{n+1}$ *(as a manifold).* (Equivalently, the latter orientation is defined at each $p \in \mathbf{R}^{n+1}$ by using the isomorphism ι_p to pull back the standard vector-space orientation of $\mathbf{R}^{n+1}$ to an orientation of the vector space $T_p\mathbf{R}^{n+1}$.) These orientations define what we will mean by "positively-oriented" and "negatively oriented" mean for bases of $\mathbf{R}^{n+1}$ and $T_p\mathbf{R}^{n+1}$.

The *standard inner product* on $\mathbf{R}^n$ is the dot-product. At each $p \in \mathbf{R}^{n+1}$, the isomorphism ι_p pulls this inner product back to an inner product on $T_p\mathbf{R}^{n+1}$. Below, these inner products on $\mathbf{R}^{n+1}$ and $T_p\mathbf{R}^{n+1}$ are intended in any reference to norms and to orthogonality.

As we have seen, the sphere $S^n := \{v \in \mathbf{R}^{n+1} : \|v\| = 1\}$ is a submanifold of $\mathbf{R}^{n+1}$. Below, for any $p \in S^n$, we regard T_pS^n as a subspace of $T_p\mathbf{R}^{n+1}$ (i.e. we implicitly identify T_pS^n with $j_{*p}(T_pS^n)$, where $j : S^n \to \mathbf{R}^{n+1}$ is the inclusion map).

(a) For each $p \in S^n$, define $N_p \in T_p\mathbf{R}^{n+1}$ by $N_p = i_p^{-1}(p)$. This is the vector that's commonly called the *outward-pointing unit normal to* S^n *at* p. Justify the words "unit" and "normal" in this terminology by checking that $\|N_p\| = 1$ and thatT_pS^n is the orthogonal complement of span(N_p).

(b) Given $p \in S^n$ and any basis $(e_1, \ldots, e_n)$ of T_pS^n, the ordered $(n+1)$-tuple $(N_p, e_1, e_2, \ldots, e_n)$ is a basis of $T_p\mathbf{R}^{n+1}$ (since $N_p \notin T_pS^n$), and hence $(\iota_p(N_p), \iota_p(e_1), \ldots, \iota_p(e_n))$ is a basis of $\mathbf{R}^{n+1}$. The *standard orientation* of T_pS^n is defined by declaring the basis$(e_1, \ldots, e_n)$ of T_pS^n to be positively oriented if and only if $(\iota_p(N_p), \iota_p(e_1), \ldots, \iota_p(e_n))$ is a positively oriented basis of $\mathbf{R}^{n+1}$ (as defined in the setup of this problem). Show that this collection of tangent-space orientations determines an orientation of S^n (i.e. that the appropriate continuity condition is satisfied).

For the remainder of this problem, let $F : S^n \to S^n$ denote the *antipodal map*, i.e the map $p \mapsto -p$. (Note that, for a point p in a general manifold, there is no such thing as "$-p$"; in defining this notation for $p \in S^n$, we are relying on the fact that S^n is a subset of a vector space.)

(c) Check that F is a smooth involution (hence a diffeomorphism) with no fixed-points.

(d) Show that F preserves orientation if n is odd, and reverses orientation if n is even.

For the remainder of this problem, let $M = S^n/\sim$, where the equivalence relation $\sim$ is the one generated by "$p \sim F(p)$", and where M is given the induced smooth structure (see problem 6(c)).

(e) Show that M is diffeomorphic to the projective space $\mathbf{R}P^n = P(\mathbf{R}^{n+1})$, **as**

defined in the first homework assignment.[3]

(f) Show that M (and therefore $\mathbf{R}P^n$) is orientable if and only if n is odd.

(g) Show that S^n "is" (more precisely, is diffeomorphic to) the orientation double-cover of $\mathbf{R}P^n$ if and only if n is even. (Part (e) shows that S^n is always *some* double-cover of $\mathbf{R}P^n$, but a general double-cover of a manifold need not be the *orientation* double-cover.)

[3]Another common definition of $\mathbf{R}P^n$ is $S^n/\sim$, of course, but that's not the definition we used. You're showing here that the two definitions yield the same manifold, up to diffeomorphism.