Sufficient Conditions for Paracompactness of Manifolds

The purpose of these notes is to examine some relations among some topological restrictions that are very often included in *definitions* of "manifold", the most common being constant-dimensionality, Hausdorffness and either second countability paracompactness. For this reason, we do *not* include any such restrictions in our definition of "manifold". In these notes, a topological manifold is simply a pair consisting of a topological space M and a C^0 maximal atlas on M.

Definition 1 Let X be a topological space.

1. Given two open covers $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}, \mathcal{V} = \{V_\beta\}_{\beta\in B}$ of X, we say that $\mathcal{V}$ *refines* (or *is a refinement of*) $\mathcal{U}$ if for all $\beta \in B$, there exists $\alpha \in A$ such that $V_\beta \subset U_\alpha$.

2. A collection of subsets $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ of X is called *locally finite* if for all $x \in X$, there exists an open neighborhood W of x intersecting U_α for only finitely many $\alpha \in A$.

3. X is *paracompact* if every open cover of X admits a locally finite refinement.

Remark 2 (1) Every subcover of an open cover $\mathcal{U}$ is a refinement of $\mathcal{U}$ (hence "compact" implies "paracompact"), but a refinement of $\mathcal{U}$ need not be a subcover of $\mathcal{U}$. (2) If a cover $\mathcal{V}$ of X is locally finite, then for all $x \in X$, only finitely many elements of $\mathcal{V}$ contain x. The converse is false (i.e. "pointwise finite" does not imply "locally finite").

Since paracompactness is not the most transparent condition in the world, and since its definition does not make it obvious how to verify that a space is paracompact, we give in these notes some conditions that suffice for paracompactness of a finite-dimensional Hausdorff manifold.

Henceforth in these notes, "manifold" always means "topological, finite-dimensional manifold". When we want to make statements about smooth manifolds, we will say so explicitly.

The three conditions we consider for a manifold M are

- M is σ-compact (see below).
- M admits a countable atlas.
- M is second countable (i.e. M has a countable basis of open sets).

We will see that for a Hausdorff manifold, these three conditions are equivalent, and that each implies paracompactness. (However, these conditions are not *necessary* for paracompactness.)

Recall that a topological space is *σ-compact* if it is a countable union of compact subsets. (Here and below, "countable" means "finite or countably infinite". Thus any compact space is σ-compact.)

Let X be a topological space. We say that a sequence $\{K_i\}_{i=1}^{\infty}$ of subsets of X *exhausts* X (or is an *exhaustion of* X) if $K_1 \subset K_2 \subset K_3 \subset \dots$ and $\bigcup_{i=1}^{\infty} K_i = X$. We make the following observations:

- If $X = \bigcup_{i=1}^{\infty} \hat{K}_i$ for some countable collection of compact sets $\hat{K}_i$, then the sequence of compact sets K_i defined by $K_i = \bigcup_{j=1}^{i} \hat{K}_i$ is an exhaustion of X. Thus a σ-compact space always admits an exhaustion by compact subsets. Conversely, an exhaustion of X by compact sets exhibits X as a countable union of compact sets. Thus a topological space is σ-compact if and only if it admits an exhaustion by compact sets.

- If $\{K_i\}_{i=1}^{\infty}$ is an exhaustion X, then any subsequence of $\{K_i\}$ exhausts X as well.

- Our definition of *exhaustion* does not require K_i to be a *proper* subset of K_{i+1}. So if $\{K_i\}$ is an exhaustion of X, and $K_i = X$ for some i, then $K_j = X$ for all $j > i$, without violating anything in the definition of "exhaustion". In particular, we do not need to modify our notation for, or definition of, "exhaustion of X by compact sets $\{K_i\}_{i=1}^{\infty}$" if X is compact and equal to one of the K_i.

Some notation we will use:

1. For sets Y, Z we write $Y - Z$ rather than $Y \setminus Z$ for the set-difference $\{y \in Y \mid y \notin Z\}$.

2. When a topological space X is understood from context, and $Z \subset X$, we write

   - $\mathrm{int}(Z)$ for the interior of Z,
   - $\overline{Z}$ for the closure of Z, and
   - Z' for the complement of Z in X.

3. $\mathbf{Q}_+$ denotes the set of positive rational numbers.

4. $\mathbf{Q}^n$ is regarded as a subset of $\mathbf{R}^n$, namely the set of points in $\mathbf{R}^n$ with rational coordinates.

5. $B_r(x) \subset \mathbf{R}^n$ denotes the open ball of radius r centered at x.

6. To denote an indexed, countable set of objects, in order to allow both finite and infinite sets, we use the notation "$\{\text{object}_i\}_{i\in I\subset \mathbf{N}}$" to indicate that the index set I may or may not be the whole set of natural numbers.

The proposition below is motivated by the example $X = \mathbf{R}^n$, K_i = closed ball of radius i centered at the origin. The reader may find it helpful to draw this picture to follow the proof.

Proposition 3 *Let X be a Hausdorff topological space for which there is an exhaustion by a collection $\{K_i\}_{i=1}^{\infty}$ of compact subsets of X with the property that $K_i \subset \text{int}(K_{i+1})$ for all $i \geq 1$. Then X is paracompact.*

Proof: Define $K_i = \emptyset$ for $i \leq 0$. Then for $i \geq 1$, let $C_i = K_i - \text{int}(K_{i-1})$ and $W_i = \text{int}(K_{i+1}) - K_{i-2}$. Each W_i is open (because in a Hausdorff space, compact sets are closed), and for $j \geq 1$, $\bigcup_{i=1}^{j} W_i = \text{int}(K_{j+1}) \supset K_j$. Hence $\{W_i\}_{i=1}^{\infty}$ is an open cover of X. Note also that $\bigcup_{i=1}^{j} C_i = K_j$, so $\bigcup_{i=1}^{\infty} C_i = X$ as well.

Now let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an arbitrary open cover of X. For $(\alpha, i) \in A \times \mathbf{N}$, define $U_{\alpha,i} = U_\alpha \bigcap W_i$. For $i \in \mathbf{N}$ define $\mathcal{U}_i = \{U_{\alpha,i}\}_{\alpha\in A}$. Then $\{U_{\alpha,i}\}_{\alpha\in A, i\in\mathbf{N}}$ is an open cover of X that refines $\mathcal{U}$, and for each $i \geq 1$, $\mathcal{U}_i$ is an open cover of W_i.

For each $i \geq 1$, C_i is a closed subset of a compact set, hence compact. In addition,

$$C_i = K_i \bigcap (\text{int}(K_{i-1}))' \subset \text{int}(K_{i+1}) \bigcap K'_{i-2} = W_i.$$

Hence the open cover $\mathcal{U}_i$ of W_i also covers C_i, so there is a finite subcollection $V_{i,1}, \ldots, V_{i,n_i}$ that covers C_i. Since the C_i cover X, the collection $\mathcal{V} = \{V_{i,j}\}_{i\in\mathbf{N}, 1\leq j\leq n_i}$ is an open cover of X and a refinement of $\mathcal{U}$. To complete the proof, it suffices to show that $\mathcal{V}$ is locally finite.

Let $x_0 \in X$, and let (i_0, j_0) be such that $x \in V_{i_0,j_0}$. Then $x \in W_{i_0}$. Observe that $W_{i_0} \bigcap W_i = \emptyset$ for $i \geq i_0 + 3$. Since $V_{i,j} \subset W_i$, none of the sets $V_{i,j}$ with $i \geq i_0 + 3$ can intersect V_{i_0,j_0}. Thus the neighborhood V_{i_0,j_0} of x_0 intersects $V_{i,j}$ for only finitely many (i, j). Therefore the cover $\mathcal{V}$ of X is locally finite. ■

Corollary 4 *For all n, $\mathbf{R}^n$ is paracompact.*

Proof: The hypotheses of Proposition 3 are satisfied if we take K_i to be the closed ball of radius i centered at the origin. ■

In fact, every metric space (hence any subset of $\mathbf{R}^n$, with the induced metric) is paracompact, but we will not give a proof here.

Below, "atlas of well-defined dimension" means that all charts in the atlas have the same dimension (guaranteed if M is connected).

Lemma 5 *Let M be a manifold with a countable atlas of well-defined dimension. Then M is σ-compact and admits an exhaustion by a collection $\{K_i\}_{i=1}^{\infty}$ of compact subsets of M with the property that $K_i \subset \mathrm{int}(K_{i+1})$ for all $i \geq 1$.*

Proof: Let $n = \dim(M)$ and let $\{(U_i, \phi_i)\}_{i \in I \subset \mathbf{N}}$ be a countable atlas on M. For $(q, r) \in \mathbf{Q}^n \times \mathbf{Q}_+$, let $V_{i,q,r} = \phi_i^{-1}(\phi_i(U_i) \bigcap B_r(q))$, and define $\phi_{i,q,r} = \phi_i|_{V_{i,q,r}}$. Then $\{(V_{i,q,r}, \phi_{i,q,r}) \mid (i, q, r) \in I \times \mathbf{Q}^n \times \mathbf{Q}_+, V_{i,q,r} \neq \emptyset\}$ is another countable atlas on M. Let $\{(W_j, \psi_j)\}_{j \in \mathbf{N}}$ be an enumeration of this atlas (given by choosing a 1-1 correspondence between $\{(i, q, r) \in I \times \mathbf{Q}^n \times \mathbf{Q}_+, V_{i,q,r} \neq \emptyset\}$ and $\mathbf{N}$).

For each $j \in \mathbf{N}$, $\psi_j(W_j)$ is an open ball of finite radius, hence of compact closure contained in $\phi_i(U_i)$ for some i. Hence the closure $\overline{W_j}$ is also compact, being the image of the compact set $\overline{\psi_j(W_j)}$ under ϕ_i^{-1} for some i.

For $i \geq 1$, let $C_i = \bigcup_{j=1}^{i} \overline{W_j}$. Then $\{C_i\}_{i=1}^{\infty}$ is an exhaustion of M by compact sets, so M is σ-compact. In general C_i will not be contained in the interior of C_{i+1}, but we will see next that there is a subsequence of $\{C_i\}$ with the desired property.

Recursively define a subsequence of $\{C_i\}$ as follows:

1. Let $K_1 = C_1$.

2. For a given $i \in \mathbf{N}$, assume $K_i = C_{j_i}$ for some $j_i \in \mathbf{N}$. The collection $\{W_j\}$ is an open cover of M, hence of K_i. Therefore we may choose a finite sub-collection $W_{i,1}, \ldots, W_{i,k_i}$ that covers K_i. Let $A_i = \bigcup_{j=1}^{k_i} W_{i,j}$, an open set in M containing K_i.

   Since A_i is a finite union of sets in the collection $\{W_j\}$, we have $A_i \subset C_{j_{i+1}}$ for some integer $j_{i+1} > j_i$. Let $K_{i+1} = C_{j_{i+1}}$. Then $K_i \subset A_i \subset \mathrm{int}(K_{i+1})$.

Any sequence $\{K_i\}$ constructed this way is a subsequence of the exhaustion $\{C_i\}$ of M, hence is an exhaustion of M by compact sets, and moreover satisfies the condition $K_i \subset \mathrm{int}(K_{i+1})$ for all $i \geq 1$. ■

Corollary 6 *Let M be a Hausdorff topological manifold with a countable atlas of well-defined dimension. Then M is σ-compact and paracompact. In particular, any connected manifold with a countable atlas is paracompact.*

Proof: Lemma 5 plus Proposition 3. ■

Part of the conclusion of Lemma 5 is that, for a Hausdorff manifold, "has a countable atlas" implies σ-compact. The converse is also true (even without assuming Hausdorffness):

Lemma 7 *Let M be a σ-compact manifold. Then M admits a countable atlas.*

Proof: Let $\{K_i\}_{i=1}^{\infty}$ be an exhaustion of M by compact sets, and let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ be an arbitrary atlas on M. For each i, the collection $\{U_\alpha\}_{\alpha \in A}$ is an open cover of K_i, so a finite subcollection $U_{\alpha(i,1)}, \ldots U_{\alpha(i,k_i)}$covers K_i. Then $\{(U_{\alpha(i,j)}, \phi_{\alpha(i,j)}) \mid 1 \leq i < \infty, 1 \leq j < k_i\}$ is a countable atlas on M. ■

Although manifolds with connected components of different dimensions are rarely of interest, for the sake of shortening the statement of Corollary 6 we show next that the "well-defined dimension" condition can be removed.

Lemma 8 *Let X be a topological space. Suppose that X is the disjoint union of open sets X_α (α running over some index set A), each of which is paracompact. Then X is paracompact.*

Proof: Let $\mathcal{U} = \{U_\beta\}_{\beta \in B}$ be an open cover of X. For $(\alpha, \beta) \in A \times B$, define $U_{\alpha,\beta} = X_\alpha \bigcap U_\beta$. For each α, the collection $\mathcal{U}_\alpha = \{U_{\alpha,\beta}\}_{\beta \in B}$ is an open cover of X_α, hence has a locally finite refinement $\{W_{\alpha,\gamma}\}_{\gamma \in C_\alpha}$. Then $\mathcal{V} := \{W_{\alpha,\gamma} \mid \alpha \in A, \gamma \in C_\alpha\}$ is a locally finite refinement of $\mathcal{U}$. ■

Corollary 9 *Let M be a Hausdorff topological manifold with a countable atlas. Then M is paracompact.*

Proof: For $n \geq 0$, let $M^n \subset M$ denote the union of all n-dimensional connected components of M. Then each M^n is an open subset of M with a countable, n-dimensional atlas. (If $\{(U_i, \phi_i)\}_{i \in I \subset \mathbf{N}}$ is a countable atlas on M, then for each n, $\{(U_i, \phi_i) \mid i \in I, U_i \bigcap M^n \neq \emptyset\}$ is a countable, n-dimensional atlas on M^n.) By Corollary 6 each M^n is paracompact. Then by Lemma 8, M is paracompact. ■

Next we consider the third condition listed on p. 1: having a countable basis of open sets.

Proposition 10 *Let M be a topological manifold. Then M has a countable basis of open sets if and only if M has a countable atlas.*

Proof: (i) Assume that M has a countable atlas $\{(U_i, \phi_i)\}_{i \in I \subset \mathbf{N}}$. For $n \geq 0$, let M^n be the union of n-dimensional connected components of M. The collection $\mathcal{V}_n$ of open sets $V_{i,q,r}$ constructed in the proof of Lemma 5, where $(i, q, r) \in I \times \mathbf{Q}^n \times \mathbf{Q}_+$, is a countable basis of the topology of M^0. Then $\bigcup_{n=0}^{\infty} \mathcal{V}_n$ is a countable basis of the topology of M.

(ii) Let $\{V_i\}_{i \in I \subset \mathbf{N}}$ be a countable basis of open sets of M. (I will necessarily be all of $\mathbf{N}$ unless $\dim(M) = 0$, but we do not need that fact.) Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ be an arbitrary atlas on M. For each α, the open set U_α is a union of sets in the basis $\{V_i\}$; thus $U_\alpha = \bigcup_{j \in J_\alpha} V_j$ for some set $J_\alpha \subset I$.

Let $J = \bigcup_{\alpha \in A} J_\alpha$. Thus $J \subset \mathbf{N}$, and for all $j \in J$ there exists $\alpha \in A$ such that $V_j \subset U_\alpha$. For each $j \in J$, select such an α and denote it $\alpha(j)$. Then $\bigcup_{j \in J} V_j = \bigcup_{\alpha \in A} U_\alpha = M$, so $\{V_j\}_{j \in J}$ is a countable open cover of M, and $\{(V_j, \phi_{\alpha(j)}|_{V_j})\}_{j \in J}$ is a countable atlas on M. ■

Assembling what we have shown, we have the following theorem:

Theorem 11 *Let M be a Hausdorff manifold. Then the following are equivalent:*

1. *M is σ-compact.*
2. *M admits a countable atlas.*
3. *M is second-countable (i.e. M has a countable basis of open sets).*

If M satisfies any of these conditions, then M is paracompact.

Proof: This follows from Corollary 6, Lemma 7, and Proposition 10. ■

Remark 12 For smooth manifolds, we have an analogous theorem, in which the hypothesis of Theorem 11 is replaced by "Let M be a smooth Hausdorff manifold", and "atlas" means "smooth atlas". To see that this is true, all we need check is that our proofs that conditions 1 and 3 in Theorem 11 imply condition 2 still work in the smooth setting. For each of the proofs "$1 \implies 2$" and "$3 \implies 2$", to obtain our eventual countable atlas, we took an arbitrary atlas, refined the open cover, and then took all our chart-maps to be restrictions of the original atlas' chart-maps to the smaller sets comprising the refinement. If M is smooth, and we start with an arbitrary smooth atlas and go through the same procedure, the overlap-maps will simply be restrictions of the original smooth overlap-maps, hence will be smooth. Thus the countable atlas we construct is smooth.

The converse of the last statement in Theorem 11 is false, in every dimension. For example, let A be an uncountable set with the discrete topology. Then $S^n \times A$, with the induced topology (thus an uncountable disjoint union of copies of the sphere S^n, each of which is a connected component of $S^n \times A$), is a paracompact Hausdorff manifold but is not σ-compact.

However, in practice, *naturally occurring* manifolds tend to be σ-compact; the author does not know a *naturally occurring* counterexample.