Differential Geometry—MTG 6256—Fall 1999
Problem Set 3

1. Prove that if $(M, \mathcal{A})$ is a connected C^k manifold and $k \geq 1$, then all charts in $\mathcal{A}$ have the same dimension.

2. (a) Show that a submanifold of a manifold is itself a manifold. More specifically, use the definition of "submanifold" to produce an atlas for the submanifold.

(b) Prove that if $F : M \to N$ is a smooth map of manifolds and $X \subset M$ is a submanifold, then $F|_X : X \to N$ (the restriction of F to X) is also a smooth map of manifolds.

3. If V is a vector space, the *projectivization* $P(V)$ is defined to be the set of lines through the origin in V, with a suitable topology. This applies whether V is a real or complex vector space; "line through the origin" means the set of real or complex multiples of a fixed nonzero vector accordingly as V is real or complex. Alternatively, $P(V) = (V - \{0\})/\sim$, where the equivalence relation $\sim$ is defined by $v \sim w \iff v = tw$ for some scalar t (real or complex, accordingly), and is topologized using the quotient topology. (You do not need to know what "quotient topology" means to do this problem, but you can read about it in the glossary.) If $V = \mathbf{R}^{n+1}$ it is convenient to write the typical element of V as $x = (x^0, x^1, \ldots x^n)$ and (if $x \neq 0$) the corresponding element of $P(V)$ as $[x^0, x^1, \ldots, x^n]$; similar notation is used if $V = \mathbf{C}^{n+1}$ but I ask that you use z instead of x in this case below. $P(\mathbf{R}^{n+1})$ is also denoted $\mathbf{R}P^n$ (*real projective space*); $P(\mathbf{C}^{n+1})$ is also denoted $\mathbf{C}P^n$ (*complex projective space*). Real and complex projective spaces can be defined other ways (as was done in class for $\mathbf{R}P^2$), but the definition in this problem set is more useful for many purposes.

(a) Let $V = \mathbf{R}^{n+1}$. For $0 \leq i \leq n$ define open sets $V_i \subset V - \{0\}$ by $V_i = \{x \in V \mid x^i \neq 0\}$. Note that if $x \in V_i$, then $tx \in V_i$ $\forall t \neq 0$, and that the collection $\{V_i\}$ covers $V - \{0\}$. Show that this cover determines a cover $\{U_i\}$ of $P(V)$ and that there is a 1-1 correspondence $\phi_i : U_i \to \mathbf{R}^n$. Show that $\{(U_i, \phi_i)\}$ is a C^∞ atlas for $P(V)$, and hence that $\mathbf{R}P^n$ is a manifold of dimension n.

(b) Analogously, show that $\mathbf{C}P^n$ is a manifold of dimension $2n$. (Note: there is such a thing as a complex manifold, and as one might guess, $\mathbf{C}P^n$ is a complex n-dimensional manifold. However, the concept is subtler than one might think, and for us "manifold" will always mean "real manifold" unless otherwise specified.)

(c) Show that $\mathbf{C}P^1$, also called the *Riemann sphere*, is diffeomorphic to S^2.

(d) For both the real and complex cases (a) and (b), show that the quotient map (or *projection*) $\pi : V - \{0\} \to P(V)$ is smooth.

(e) For $V = \mathbf{C^{n+1}} \cong_{\mathbf{R}} \mathbf{R^{2n+2}}$, let H be the restriction of the projection π to the unit sphere $S^{2n+1} \subset \mathbf{R}^{2n+2}$. Show that H is surjective and smooth. In view of (c) and the Hopf map defined in class, $H : S^{2n+1} \to \mathbf{C}P^n$ is called the *generalized Hopf map.* (Note: there is a reason Problem 2 was given before Problem 3.)

4. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be smooth. Prove that the inverse image of a regular value is either empty or a submanifold of codimension m (i.e. dimension $n - m$).

5. (Application of problem 4.) Let $\mathrm{Sym}_n(\mathbf{R}) \subset M_n(\mathbf{R}) \cong \mathbf{R}^{n^2}$ be the subspace consisting of $n \times n$ symmetric matrices. Define $F : M_n(\mathbf{R}) \to \mathrm{Sym}_n(\mathbf{R})$ by $F(A) = A^t A$. (In Problem Set 1 we computed DF, but we did not observe at the time that the image of F, and hence of $D_A F$ for all A, lies in the subspace $\mathrm{Sym}_n(\mathbf{R})$.) Let $I \in M_n(\mathbf{R})$ be the identity; note that $F^{-1}(I) = \{A \in M_n(\mathbf{R}) \mid A^t A = I\}$, which is also known as the *orthogonal group* $O(n)$. Show that I is a regular value of F, and hence that $O(n)$ is a submanifold of $M_n(\mathbf{R})$. What is the dimension of $O(n)$?

Note: $O(n)$ is not connected; it has two connected components, the set $SO(n)$ of orthogonal matrices of determinant 1, and the set of orthogonal matrices of determinant -1. This example shows that non-connected manifolds can arise naturally in important examples.

6. The *Grassmannian* or *Grassmann manifold* $G_k(\mathbf{R}^n)$ is defined to be the set k-dimensional subspaces of $\mathbf{R}^n$. (This is a generalization of projective space; $G_1(\mathbf{R}^n) = \mathbf{R}P^{n-1}$.) To construct an atlas, observe that given any k-plane X through the origin, any sufficiently close k-plane Y through the origin is the graph of a unique linear map $T : X \to X^\perp$, where $X^\perp$ is the orthogonal complement of X. (Here "sufficiently close" means that $Y \bigcap X^\perp = \{0\}$.) For each k-element subset $I = \{i_1, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$, let X_I be the subspace consisting of all $x \in \mathbf{R}^n$ all of whose coordinates other than those in positions $i_1, \ldots, i_k$ vanish. Let $V_I \subset \mathbf{R}^n$ be the complement of $X_I^\perp$.

(a) Show that $\{V_I\}$ is an open cover of $\mathbf{R}^n - \{0\}$ and determines a cover $\{U_I\}$ of $G_k(\mathbf{R}^n)$, analogously to Problem 3a.

(b) Show that there is a 1-1 correspondence ϕ_I from U_I to the set of linear maps $T : X_I \to X_I^\perp$, hence with the set of $(n-k) \times k$ matrices, hence with $\mathbf{R}^{k(n-k)}$.

(c) Show that the overlap maps $\phi_J \circ \phi_I^{-1}$ are smooth, and hence that $G_k(\mathbf{R}^n)$ is a manifold of dimension $k(n-k)$.