Differential Geometry—MTG 6256—Fall 1999
Problem Set 4

1. Let M be an n-dimensional manifold, let (U, ϕ) and (V, ψ) be charts of M, and let $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$ be the associated charts of the tangent bundle TM. Let J be the $n \times n$ Jacobian matrix of $\psi \circ \phi^{-1}$ and let $\tilde{J}$ be the $2n \times 2n$ Jacobian matrix of $\tilde{\psi} \circ \tilde{\phi}^{-1}$. Express $\tilde{J}$ in terms of J.

2. Let M be a manifold and let $\pi : TM \to M$ be the natural projection (carrying a tangent vector to the point at which the vector is based). Show in the following two ways that π is a submersion: (a) by using problem 1; (b) by using the Chain Rule for maps of manifolds, simple linear algebra, and the fact that $\pi \circ s =$ identity, where $s : M \to TM$ is the map sending a point p to the zero vector in T_pM.

3. Let M, N be manifolds, $F : M \to N$ a submersion, $q \in \mathrm{Image}(F) \subset N$, $Z = F^{-1}(q)$ (a submanifold of codimension $\dim(N)$, by the Regular Value Theorem), and $p \in Z$. Prove that $T_pZ = \ker(F_{*p})$. (More precisely, letting $\iota : Z \to M$ be the inclusion map of the subset Z, you are proving that $\iota_{*p}(T_pZ) = \ker(F_{*\iota(p)})$. If you did not understand this last remark, don't worry.)

4. Let $\phi : S^2 \to \mathbf{R}^2$ be sterographic projection through the north pole, let $\iota : S^2 \to \mathbf{R}^3$ be the usual embedding of the sphere in $\mathbf{R}^3$, and let $F = \iota \circ \phi^{-1}$. (In other words, F is the inverse of stereographic projection, but viewed as a map $\mathbf{R}^2 \to \mathbf{R}^3$ [whose image happens to lie in S^2] rather than as a map $\mathbf{R}^2 \to S^2$.) Compute F_*.

5. Let M be a manifold, X and Y vector fields on M, and $[X, Y]$ the commutator of X and Y (the differential operator on functions on M defined by $[X, Y](f) = X(Y(f)) - Y(X(f))$.)

(a) Without using any facts about Lie derivatives, show directly that $[X, Y]$ is Leibnizian linear functional, and therefore a vector field.

(b) In the local coordinates $\{x^i\}$ given by some chart, suppose that $X = f^i \frac{\partial}{\partial x^i}$ and $Y = g^i \frac{\partial}{\partial x^i}$. Compute the local-coordinate expression for $[X, Y]$.

6. Let M be a manifold, X a vector field on M, Φ the flow of X. Prove that fixed-points of the flow—i.e. those points $p \in M$ for which $\Phi_t(p) = p\ \forall t$—are exactly those p at which $X_p = 0$.

7. Notation as in problem 6. Recall that an *integral curve* of X is a "Φ-orbit", i.e. a set of the form $\{\Phi_t(p)\}$ with p fixed and t varying. Prove that multiplying X by a nonzero function only reparametrizes the integral curves; it does not change the underlying point-sets. More precisely, if $Y = fX$ for some nonzero function f, prove that every integral curve of Y is an integral curve of X and vice-versa.

8. Let M be a manifold, X a vector field on M, and suppose that $X_p \neq 0$ at some $p \in M$. Prove that there are local coordinates $\{x^i\}$ on some open neighborhood U of p such that $X = \partial/\partial x^1$ on U. (Hint: use the flow of X.)

9. The integral curves of a nonzero vector field on a manifold M are one-dimensional submanifolds of M. Given *two* vector fields X and Y, one can ask whether their integral curves "hang together" to produce two-dimensional submanifolds (at least locally) the points of any one of which can be connected to each other by moving along piecewise smooth curves each segment of which is a portion of an integral curve of X or Y. This problem gives a sufficient condition on X and Y for them to generate a two-dimensional submanifold locally in this way. The condition is also necessary in a certain sense (see part (c)). A more general sufficient condition is given in part (d). (Note: Each part of this problem from (b) on depends on part (a). If you can't do part (a), you may assume it in order to try the later parts. If you're unable to do this problem now, you can turn it in later in the semester as your facility with the subject develops, but the proofs involve no ingredients that you don't already have.)

Let M be a manifold, X and Y vector fields on M.

(a) Prove that $[X, Y] \equiv 0$ iff the flows of X and Y commute, i.e. if $\Phi_s \circ \Psi_t(p) = \Psi_t \circ \Phi_s(p)$ whenever both sides are defined. (One direction of the "iff" is much easier than the other. The hard direction is what's needed below.)

(b) (Generalization of problem 8.) Suppose that X and Y are linearly independent at p, and that $[X, Y] \equiv 0$. Prove that there are local coordinates $\{x^i\}$ on some open neighborhood U of p such that $X = \partial/\partial x^1$ and $Y = \partial/\partial x^2$ on U. (Hint: use the flows.)

(c) Hypotheses as in (b). Prove that there exists a two-dimensional submanifold L of M, containing p, such that at each point q of L the tangent space T_qL is spanned by X_q and Y_q. Conversely, show that given any two-dimensional submanifold L of M, there exist vector fields X, Y defined on a neighborhood of L satisfying the conditions in part (b).

(d) Suppose that on some open set V, X_q and Y_q are linearly independent at each $q \in V$ and that $[X, Y]_q$ lies in the span of X_q and Y_q (so there are functions f and g such that $[X, Y] = fX + gY$ for some functions f and g; it follows from part (b) that f and g are smooth). Prove for each $p \in V$ there exist a neighborhood U of p and locally-defined vector fields $\tilde{X}, \tilde{Y}$ with the same span as X and Y at each point of U, satisfying $[\tilde{X}, \tilde{Y}] \equiv 0$. (Hence, from part (c) these conditions on X and Y can replace the less general conditions in part (b).)