Differential Geometry—MTG 6256—Fall 1999
Problem Set 5

1. Let $\{v_i\}, \{w_j\}$ be bases of the finite-dimensional vector spaces V, W respectively. Let $A : V \to W$ be a linear map and let $\hat{A}$ be the matrix of A with respect to the given bases. Show that, with respect to the bases dual to $\{v_i\}, \{w_j\}$, the matrix of the adjoint map $A^* : W^* \to V^*$ is the transpose of $\hat{A}$.

2. (a) Let X be a vector field on a manifold M. Show that X is smooth (as a map from M to TM) iff when X is expressed in local coordinates as $f^i \frac{\partial}{\partial x^i}$, the coefficient functions f^i are smooth.

(b) Let θ be a 1-form on M. Show θ is smooth (as a map from M to T^*M) iff when θ is expressed in local coordinates as $g_i dx^i$, the coefficient functions g_i are smooth.

(c) Generalize (a) and (b) to sections of arbitrary vector bundles over manifolds.

3. Prove that a line bundle L admits a nonvanishing section iff L is a trivial bundle.

4. Let M be a manifold and let $F : M \to M$ be a diffeomorphism. Let A, B be tensor fields and X a vector field on M.

(a) Show that $F^*(A \otimes B) = F^*A \otimes F^*B$.

(b) Show that $\mathcal{L}_X(A \otimes B) = (\mathcal{L}_X A) \otimes B + A \otimes \mathcal{L}_X B$.

(c) Assume that the tensor bundles of which A and B are sections are such that there is a contraction operation $\lrcorner$ defined. Show that $F^*(A \lrcorner B) = F^*A \lrcorner F^*B$.

(d) Assumptions as in (c). Show that $\mathcal{L}_X(A \lrcorner B) = (\mathcal{L}_X A) \lrcorner B + A \lrcorner \mathcal{L}_X B$.