Differential Geometry—MTG 6256—Fall 1999
Problem Set 7

1. Let X be a vector field on a manifold M. Prove that for all $p \geq 0$ and all $\omega \in \Omega^p(M)$,

$$d(\mathcal{L}_X(\omega)) = \mathcal{L}_X(d\omega).$$

(You may use the result of Problem 2 on Problem Set 6.)

2. Prove the following:
   (a) (closed form)$\wedge$(closed form) = (closed form).
   (b) (closed form)$\wedge$(exact form) = (exact form) =(exact form)$\wedge$(closed form).

Using parts (a) and (b),
   (c) Show that wedge product of forms induces a well-defined product "$\cup$" on cohomology via the formula

$$[\omega] \cup [\eta] = [\omega \wedge \eta].$$

(Here the brackets denote cohomology class of the differential form within.) Thus the de Rham $H^*(M)$ has both an additive and a multiplicative structure, satisfying the usual distributive and associative laws, so we often refer to $H^*(M)$ as the cohomology *ring*.

3. Let M, N be manifolds, and let $\pi_M : M \times N \to M, \pi_N : M \times N \to N$ be the projections onto the first and second factors respectively. In class we saw that any map of manifolds induces a map on cohomology via pullback of forms, so in this case we have maps $\pi_M^* : H^p(M) \to H^p(M \times N)$ and $\pi_N^* : H^p(N) \to H^p(M \times N)$ for all $p \geq 0$. From Problem 2, we therefore have maps

$$\begin{array}{rcl} H^p(M) \times H^q(N) & \to & H^{p+q}(M \times N) \\ [\omega] \times [\eta] & \mapsto & \pi_M^*[\omega] \cup \pi_N^*[\eta] = [(\pi_M^*\omega) \wedge \pi_N^*\eta]. \end{array}$$

The maps above are linear (over $\mathbf{R}$) in each factor and hence define maps $H^p(M) \otimes H^q(N) \to H^{p+q}(M \times N)$ by the same formula with $\times$ replaced by $\otimes$ on the left-hand side. We can collate these maps for a fixed value of $p + q$ and obtain maps

$$K : \bigoplus_{p+q=r} H^p(M) \otimes H^q(N) \to H^r(M \times N).$$

(Note that p or q can be zero in this sum.) The *Künneth formula*, which we will not prove, asserts that the maps K are isomorphisms. This is sometimes written more succinctly as

$$H^*(M \times N) \cong H^*(M) \otimes H^*(N),$$

an isomorphism of graded rings. (This actually says more than that the each map K is an isomorphism, since a ring isomorphism implies an equivalence of product structures, not just linear structures.)

Use the Künneth formula to compute the cohomology ring of each of the following spaces. In each case, give the generators of the ring in terms of the generators of the cohomology of each factor. (For example, $H^*(S^n)$ has a generator "1" in degree 0 [the constant function 1], a generator "a" in degree n [corresponding to an n-form with total integral 1, or any other nonzero number]. When $n = 1$ one can take a to be the cohomology class of the form we called "$d\theta$" in class.)

(a) $S^1 \times S^1$.

(b) $S^m \times S^n$.

(c) $S^1 \times S^1 \times \ldots \times S^1$ (n factors).

4. Compute the standard metric on the sphere S^n (the metric induced by the embedding $S^n \hookrightarrow \mathbf{R}^{n+1}$) in coordinates given by stereographic projection. If you do this correctly, you should find a metric conformally equivalent to the standard metric on $\mathbf{R}^n$ (see Problem 6).

5. Let V be a finite-dimensional vector space with a nondegenerate quadratic form h (i.e. an "inner product" that may or may not be positive-definite, such as a Riemannian or Lorentzian metric). In class we saw that h determines an isomorphism $\mathfrak{h} : V \to V^*$ and a nondegenerate quadratic form $\hat{h}$ on V^*. Let $\{e_i\}$ be an arbitrary basis of V and let $\{\theta^i\}$ be the dual basis of V^*; define components $h_{ij}, \hat{h}^{ij}$ by $h = h_{ij}\theta_i \otimes \theta_j$ and $\hat{h} = \hat{h}^{ij} e_i i \otimes e_j$. Let $h_{..}, \hat{h}^{..}$ denote the matrices with components $h_{ij}, \hat{h}^{ij}$ respectively, and let $\tilde{h}^{..}$ denote the inverse of the matrix $h_{..}$.

(a) Show that $\mathfrak{h}(e_i) = h_{ij}\theta^j$ and $\mathfrak{h}^{-1}(\theta^i) = \tilde{h}^{ij} e_j$.

(b) For $v \in V$ let $\{v^i\}$ denote the components of v in the basis $\{e_i\}$. Show that if we define $v_i = h_{ij}v^j$, then $\mathfrak{h}(v^i e_i) = v_j\theta^j$.

(c) Show that if $v, w \in V$, then $h(v, w) = v_i w^i$.

(d) Show that $\hat{h}^{..} = \tilde{h}^{..}$.

6. Two inner products g_1, g_2 on a vector space V are said to be *conformally related* or *conformally equivalent* if one is a scalar multiple of the other: $g_2 = c^2 g_1$ for some $c > 0$. (We use c^2 rather than c so that the norms are related by a factor of c rather than $c^{1/2}$.) Analogously, two Riemannian metrics g_1, g_2 on a manifold M are called conformally equivalent if at each point of M the inner products on T_pM are conformally equivalent, i.e. if there exists a positive *function* $f : M \to \mathbf{R}$ such that $g_2 = f^2 g_1$.

Let g_1, g_2 be conformally related metrics on M with conformal factor f as above. For purposes of this problem, let $g_1^{(p)}, g_2^{(p)}$ be the induced metrics on p-forms.

(a) Let $\{e_i\}$ be a local g_1-orthonormal basis of TM and let $\{\theta^i\}$ be the dual basis. By what powers of f must one multiply $\{e_i\}$ and $\{\theta_i\}$ to get g_2-orthonormal bases?

(b) Find the formula relating the induced metrics on 1-forms (i.e. find the exponent

m for which $g_2^{(p)} = f^m g_1^{(p)}$.

(c) More generally, find the exponent m for which $g_2^{(p)} = f^m g_1^{(p)}$.

(d) Suppose that the manifold M above is oriented and let $n = \dim(M)$. Let $*_{p,i} : \Omega^p(M) \to \Omega^{n-p}(M)$ be the Hodge star-operator on p-forms for the metric g_i, $i = 1, 2$. Find the relation between $*_{p,2}$ and $*_{p,1}$ (your answer should involve n, p, and f). Using your answer, explain the following statement: "The star operator is conformally invariant on forms of the middle degree."

Remark. Clearly if two inner products on a vector space V are conformally equivalent, then the angles they determine between vectors are the same. The converse is also true. I suggest you prove this as an exercise, but you need not hand it in.

7. Let (M, g) be an oriented Riemannian manifold of dimension n.

(a) In terms of n and p, figure out the sign in $* \circ * = \pm \mathrm{Id} : \Omega^p(M) \to \Omega^p(M)$.

(b) In terms of n and p, figure out the sign in $d^* = \pm * \circ d \circ \star : \Omega^p(M) \to \Omega^{p-1}(M)$.

8. Let V be an n-dimensional oriented vector space with an inner product. We can define an operator $* : \bigwedge^p(V) \to \bigwedge^{n-p}(V)$ by repeating the construction for manifolds, merely restricting to the fiber over a point (i.e. the analogy is $V = T_q^* M$ for some $q \in M$).

(a) Show that the following map from the set of simple elements to the Grassmannian,

$$v_1 \wedge \ldots \wedge v_p \mapsto \mathrm{span}\{v_1, \ldots v_p\},$$

is well-defined. (I.e. if ω is expressible as a simple element in two different ways, then the p-planes determined as above above are the same.)

(b) Show that if ω is simple, then so is $*\omega$.

(c) What is the the geometric relation between the images of ω and $*\omega$ under the map in part (a)?