**Differential Geometry—Fall 2014**
**Assignment 2's non-book problems**

1. Let $x, y, z$ be the usual coordinate-functions on $\mathbf{R}^3$, let $U = \mathbf{R}^3 \setminus \{(0,0,0)\}$, define $r = \sqrt{x^2+y^2+z^2}$, and let $\omega$ be the 2-form on $U$ given by

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3}\ .$$

Compute $d\omega$.

2. **Differential forms and Electricity & Magnetism.** [Note: you do not need to know any physics to do this problem. It will just have additional meaning for you if you *do* know the relevant physics.]

For this problem, index the standard coordinate functions on $\mathbf{R}^4$ from 0 to 3 rather than from 1 to 4, and define $t = x^0, x = x^1, y = x^2, z = x^3$ (so that the coordinate-functions can be written either as $t, x, y, z$ or as $x^0, x^1, x^2, x^3$, whichever is more convenient for a given purpose). We will refer to the coordinate-function $t$ as *time*, and the other three coordinates as *spatial* coordinates.

Let $U \subset \mathbf{R}^4$ be open. The *electric* and *magnetic fields* on $U$ are $\mathbf{R}^3$-valued functions $\mathbf{E}, \mathbf{B}$. In Calculus 3 terminology, $\mathbf{E}$ and $\mathbf{B}$ are vector fields ($\mathbf{E} = E_1\mathbf{i} + E_2\mathbf{j} + E_3\mathbf{k}$, etc. for $\mathbf{B}$), but are allowed to depend on time as well as on spatial coordinates. *Maxwell's equations in vacuum*, in units in which the speed of light is 1, are collectively the following set of four equations:

$$\nabla \cdot \mathbf{E} = 0 \qquad\qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\tfrac{\partial \mathbf{B}}{\partial t} \qquad\qquad \nabla \times \mathbf{B} = \tfrac{\partial \mathbf{E}}{\partial t}$$

Define a 2-form $F$ on $U$ by

$$F = dt \wedge (E_1 dx + E_2 dy + E_3 dz) - B_1 dy \wedge dz - B_2 dz \wedge dx - B_3 dx \wedge dy.$$

At each point $p \in U$, define a linear operator $\star_M : \bigwedge^2 T_p^*\mathbf{R}^4 \to \bigwedge^2 T_p^*\mathbf{R}^4$ by defining it as follows on elements of the standard basis:

$$\star_M(dt \wedge dx) = -dy \wedge dz, \quad \star_M(dt \wedge dy) = -dz \wedge dx, \ \star_M(dt \wedge dz) = -dx \wedge dy,$$

$$\star_M(dx \wedge dy) = dt \wedge dz, \quad \star_M(dy \wedge dz) = dt \wedge dx, \ \star_M(dz \wedge dx) = dt \wedge dy.$$

(There is a general definition of "star operators" such as the one above and the one in your previous homework assignment. We are not far enough along in this class for the general definition. I've inserted the subscript "$M$" above because this operator on bi-covectors turns is the one appropriate to Minkowski space—for those students who know what Minkowski space is—rather than Euclidean space.) Extend the operator $\star_M$ pointwise to a linear map $\star_M : \Omega^2(U) \to \Omega^2(U)$.

(a) Show that Maxwell's equations in vacuum are equivalent to the following pair of equations:

$$dF = 0, \qquad d(\star_M F) = 0. \tag{0.1}$$

(b) The (electromagnetic) *scalar potential* and *vector potential* on $U$ are, respectively, a real-valued function $\Phi$ and an $\mathbf{R}^3$-valued function $\mathbf{A} = (A_1, A_2, A_3)$ on $U$. Define a 1-form $\mathcal{A}$ on $U$ by

$$\mathcal{A} = \Phi dt - A_1 dx - A_2 dy - A_3 dz.$$

Let $F$ be as in part (a). Show that the equation $F = d\mathcal{A}$ is equivalent to the pair of equations

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

(c) The *charge density* and *current density* on $U$ are, respectively, a real-valued function $\rho$ and an $\mathbf{R}^3$-valued function $\mathbf{J} = (J_1, J_2, J_3)$ on $U$. Define a 1-form $\mathcal{J}$ on $U$ by

$$\mathcal{J} = \rho dt - J_1 dx - J_2 dy - J_3 dz.$$

At each point $p \in U$, define a linear operator $\star_M : T_p^*\mathbf{R}^4 \to \bigwedge^3 T_p^*\mathbf{R}^4$ by defining it as follows on elements of the standard basis:

$$\star_M(dt) = dx \wedge dy \wedge dz,$$
$$\star_M(dx) = dt \wedge dy \wedge dz, \ \star_M(dy) = dt \wedge dz \wedge dx, \ \star_M(dz) = dt \wedge dx \wedge dy.$$

Extend this operator $\star_M$ pointwise to a linear map $\star_M : \Omega^1(U) \to \Omega^3(U)$.

*Maxwell's equations (at the microscopic level) in the presence of sources* are the following modified version of Maxwell's equations in vacuum:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\tfrac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = 4\pi\mathbf{J} + \tfrac{\partial \mathbf{E}}{\partial t}.$$

Show that this set of four equations is equivalent to the pair of equations

$$dF = 0, \qquad d(\star_M F) = \star_M(4\pi\mathcal{J}). \tag{0.2}$$

(d) Note that the second equation in (0.2) implies $d(\star_M \mathcal{J}) = \frac{1}{4\pi} dd(\star_M F) = 0$. Show that the equation $d(\star_M \mathcal{J}) = 0$ is equivalent to the equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{0.3}$$

Equation (0.3), known in physics as the *continuity equation*, expresses conservation of charge.