Differential Geometry—MTG 6256—Fall 2014
Assignment 4's non-book problems

Note: Problems 1 and 4 could have been asked on Assignment 3, but your professor did not want to make that assignment any lengthier than it already was.

1. Let $\mathrm{Sym}_n(\mathbf{R}) \subset M_n(\mathbf{R}) \cong \mathbf{R}^{n^2}$ be the subspace consisting of $n \times n$ symmetric matrices (those A for which $A^t = A$). Define $F : M_n(\mathbf{R}) \to \mathrm{Sym}_n(\mathbf{R})$ by $F(A) = A^t A$. (In Problem Set 1 we computed DF, but we did not observe at the time that the image of F, and hence of $D_A F$ for all A, lies in the subspace $\mathrm{Sym}_n(\mathbf{R})$.) Let $I \in M_n(\mathbf{R})$ be the identity; note that $F^{-1}(I) = \{A \in M_n(\mathbf{R}) \mid A^t A = I\}$, which is also known as the *orthogonal group* $O(n)$. Show that I is a regular value of F, and hence that $O(n)$ is a submanifold of $M_n(\mathbf{R})$. What is the dimension of $O(n)$?

Note: $O(n)$ is not connected; it has two connected components, the set $SO(n)$ of orthogonal matrices of determinant 1, and the set of orthogonal matrices of determinant -1. This example shows that non-connected manifolds can arise naturally in important examples.

2. Let ω be a non-vanishing n-form on a connected, oriented n-dimensional manifold M. Prove that ω is either positive at all points of M or negative at all points of M.

3. Let M and N be manifolds of dimension m and n respectively. Recall that $M \times N$ naturally inherits the structure of an $(m+n)$-dimensional manifold: if $\mathcal{A}_M = \{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}, \mathcal{A}_N = \{(V_\beta, \phi_\beta)\}_{\beta\in B}$, are atlases for M, N respectively, then $\mathcal{A}_M \times \mathcal{A}_N \underset{\text{def}}{=} \{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)\}_{(\alpha,\beta)}$ is an atlas for $M \times N$. If M and N are oriented, then $M \times N$ inherits an orientation (the *product orientation*): the orientation class of $\mathcal{A}_M \times \mathcal{A}_N$, where $\mathcal{A}_M, \mathcal{A}_N$ are arbitrary atlases of M, N within the given orientation classes.[1]

Let $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ be the projections onto the first and second factors, respectively. Let $\omega \in \Omega_c^m(M)$, $\eta \in \Omega_c^n(N)$. Prove that $\pi_M^*\omega \ \wedge \ \pi_N^*\eta \in \Omega_c^{m+n}(M \times N)$ and that

$$\int_{M\times N} \pi_M^*\omega \ \wedge \ \pi_N^*\eta = \left(\int_M \omega\right) \times \left(\int_N \eta\right),$$

where $M \times N$ is given the product orientation. (This is a generalization of the following Calculus 3 fact: If $f, g : \mathbf{R} \to \mathbf{R}$ are supported in the intervals $[a, b], [c, d]$ respectively, then $\int\int_{[a,b]\times[c,d]} f(x)g(y)dxdy = \left(\int_a^b f(x)dx\right) \times \left(\int_c^d g(y)dy\right)$.)

[1] Alternatively: if V and W are oriented vector spaces of dimension m and n respectively, the *product orientation* on $V \oplus W$ is the one for which $\{(e_1, 0), \ldots (e_m, 0), (0, e_1'), \ldots, (0, e_n')\}$ is positively oriented, where $\{e_1, \ldots e_m\}, \{e_1', \ldots e_n'\}$ are positively oriented bases of V, W respectively. Recalling that $T_{(p,q)}M \times N$ is canonically isomorphic to $T_pM \oplus T_qN$, the product orientation on $M \times N$ is the one for which (for all $(p, q) \in M \times N$), the orientation of $T_{(p,q)}M \times N$ is the product orientation determined by the orientations of the oriented vector spaces T_pM, T_qN.

The multi-part problem below is optional. If it doesn't interest you, or you don't have time, don't worry about doing it. It deals with a very important concept and tool in differential topology, and the last part of it is essential to the differential-topological definition and interpretation of the *degree* of a smooth map from one compact n-dimensional manifold to another.

4. **Transversality.** *Notation:* Given two vector subspaces U, V of a vector space W, we define their *sum* $U+V$ to be the subspace $\{u+v \mid u \in U, v \in V\}$ (also called $\text{span}\{U,V\}$).[2]

Two submanifolds M and Z of a manifold N are said to intersect *transversely* at a point $z \in N$ if $T_zM + T_zZ = T_zN$ (more precisely, if $\iota_{*z}(T_zM) + \jmath_{*z}(T_zZ) = T_zN$, where $\iota, \jmath$ are the inclusion maps of M, Z, respectively, into N). If this condition is met at all points of $M \bigcap Z$ we say simply that M and Z *intersect transversely*, or have *transverse intersection*, or that *the intersection is transverse*, and write $M \pitchfork Z$.

More generally, given manifolds M, N and a submanifold $Z \subset N$, a map $F : M \to N$ is said to be *transverse to* Z if for all $(p,z) \in M \times Z$ with $F(p) = z$, we have $F_{*p}(T_pM) + T_zZ = T_zN$. Short-hand notation for "F is transverse to Z" is "$F \pitchfork Z$". We may view this as a generalization of the definition in the previous paragraph, since in the case of two submanifolds M, Z of N, the submanifolds intersect transversely if and only if the inclusion map $\iota : M \to N$ is transverse to Z. (It's clear that this relation is symmetric in M, Z.) Note that in this case, $\iota^{-1}(Z) = M \bigcap Z$.

Transversality comes into play when we ask the question "Is the intersection of two submanifolds a submanifold?" The answer is no in general, but yes if the intersection is transverse. Transversality is a *sufficient*, but not necessary, condition for the intersection to be a submanifold. Some examples with $N = \mathbf{R}^3$, with coordinates x, y, z: (i) the submanifolds $Z = xy$-plane, $M = yz$-plane, intersect transversely; (ii) $Z = xy$-plane, $M = z$-axis, intersect transversely; (iii) $Z = x$-axis, $M = y$-axis, do not intersect transversely ; (iv) $Z = xy$-plane, $M = \{\text{graph of } z = x^2 - y^2\}$, do not intersect transversely (because of what happens at the origin).

(a) Let $N = \mathbf{R}^n$, $0 \le k \le n$, and view N as $\mathbf{R}^k \times \mathbf{R}^{n-k}$. (For the cases $k=0$ and $k=n$, the convention is $\mathbf{R}^0 = \{0\}$ and we make the obvious identifications of $\{0\} \times \mathbf{R}^n$ and $\mathbf{R}^n \times \{0\}$ with $\mathbf{R}^n$.) Let Z be the k-dimensional submanifold $\mathbf{R}^k \times \{0 \in \mathbf{R}^{n-k}\}$. Prove that if M is a manifold and $F : M \to N$ is transverse to Z, then $F^{-1}(Z)$ is a submanifold of M. (*Hint*: Consider the map $G = \pi \circ F : M \to \mathbf{R}^{n-k}$, where $\pi : \mathbf{R}^k \times \mathbf{R}^{n-k} \to \mathbf{R}^{n-k}$ is projection onto the second factor.)

(b) Use the result of part (a) to prove that if M, N are arbitrary manifolds and

[2] Note that U and V are allowed to have nontrivial intersection. When the intersection is trivial, i.e. $U \bigcap V = \{0\}$, we say that W is the *direct* sum of U and V, and (sometimes) write $W = U \oplus V$. However, we also use the symbol "$\oplus$" for the direct sum of two arbitrary vector spaces that aren't given to us as subspaces of a third.

$F : M \to N$ is transverse to a submanifold $Z \subset N$, then $F^{-1}(Z)$ is a submanifold of M. (Note that the case $Z = \{\text{point}\}$ is the Regular Value Theorem, so the theorem you're asked to prove here may be considered a generalization.) What are the dimension and codimension of $F^{-1}(Z)$?

(c) Part (b), applied to the case in which F is the inclusion map of a submanifold $M \subset N$, shows that if $M \pitchfork Z$ and $M \bigcap Z \neq \emptyset$, then $M \bigcap Z$ is a submanifold of M. For $p \in M \bigcap Z$, express $T_p(M \bigcap Z)$ in terms of T_pM and T_pZ.

(d) In the setting of part (c), $M \bigcap Z$ is also a submanifold of Z, by symmetry. It is easy to show that a submanifold of a submanifold of N is a submanifold of N, so:

- M is a submanifold of N, of a certain codimension;
- Z is a submanifold of N, of a certain codimension;
- $M \bigcap Z$ is a submanifold of M, of a certain codimension;
- $M \bigcap Z$ is a submanifold of Z, of a certain codimension; and
- $M \bigcap Z$ is a submanifold of N, of a certain codimension.

Express the last three codimensions on this list in terms of the first two. To understand what these relations are saying, after you figure out the formulas, write them out without choosing letters to represent dimensions or codimensions; i.e. using the terms "codimension of M in N", "codimension of $M \bigcap Z$ in M", etc. Try to formulate a general principle that explains (not necessarily rigorously) your findings.

(e) Independent of the earlier parts of this problem, what is a necessary and sufficient condition that a subset S of a given manifold be a zero-dimensional submanifold? (The condition should involve nothing more than point-set topology.) Apply this condition when M, Z are transversely-intersecting submanifolds of N of complementary dimensions ($\dim(M) + \dim(Z) = \dim(N)$). What do you conclude about $M \bigcap Z$ in this case? If both M and Z are compact, what stronger conclusion can you reach?