Differential Geometry 2—MAT 4930 —Spring 2015
Assignment 1

Note: Some of the problems on this assignment are simple applications of Stokes's Theorem, and could have been assigned last semester had there been time.

1. Let D be a domain with regular boundary in an oriented n-dimensional manifold M, where $n \geq 1$ and let ∂D have the induced orientation. Let $\omega \in \Omega^j(N), \eta \in \Omega^k(N)$, where $j+k=n-1$, and assume that at least one of the sets $\mathrm{supp}(\omega) \bigcap D$, $\mathrm{supp}(\eta) \bigcap D$, is compact. (Note that the compact-support assumption is superfluous if we assume that M is compact or that D is compact.) Prove the "integration-by-parts" formula

$$\int_D d\omega \wedge \eta = \int_{\partial D} \omega \wedge \eta - (-1)^j \int_D \omega \wedge d\eta.$$

Remark. The case $D=M$ (hence $\partial D = \emptyset$) is important all by itself.

2. Let M be an n-dimensional manifold. Recall that $\omega \in \Omega^k(M)$ is called *closed* if $d\omega = 0$, and *exact* if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$.

(a) Assume that M is compact, connected, and oriented, and let ω be a nonvanishing (i.e. nowhere-zero) n-form on M. Show that $\int_M \omega \neq 0$. Show also that the same conclusion can be reached if we drop the connectedness assumption, but assume that ω is positive or negative (i.e. that ω has the same sign, relative to the given orientation, at each point of M).

(b) Let $\omega \in \Omega^k(M)$. Show that if ω is exact, then for every compact oriented k-dimensional submanifold $Z \subset M$, $\int_Z \omega = 0$.

(c) Let g be a Riemannian metric on M and assume that M is compact and oriented. Let ω be the corresponding volume form. Since every n-form on M is closed (for trivial reasons), ω is closed. Show that ω is not exact.

Remark. Recall that every exact form is closed (since $d^2=0$). Thus, one necessary condition for a differential form to be exact is that it be closed. (This is a generalization of the MAP 2302 exactness test for a differential $M(x,y)dx+N(x,y)dy$, and of the Calculus 3 "curl test" for whether a vector field on an open set in $\mathbf{R}^3$ could be the gradient of some function.) Recall also that every n-form on M is closed, for trivial reasons. Thus, part (c) shows that a closed k-form need not be exact. (Part (c) illustrates this only for $k=n$, but the fact is true more generally.) Part (b) gives an exactness test for forms ω that pass the "Is $d\omega = 0$?" test: if we can find a compact, oriented k-dimensional submanifold Z for which $\int_Z \omega \neq 0$, then ω is not exact. In practice, this is the only practical tool for showing that a given closed differential form is not exact.

3. **Gradient of a function on a Riemannian manifold.** Let (M, g) be a Riemannian manifold. Recall that at each point $p \in M$, the inner product g_p induces an isomorphism $\mathsf{g}_p : T_pM \to T_p^*M$.[1] Letting the point p vary, we obtain an isomorphism $\mathsf{g} : \{\text{vector fields on } M\} \to \Omega^1(M)$, and hence an isomorphism $\mathsf{g}^{-1} : \Omega^1(M) \to \{\text{vector fields on } M\}$. The *gradient* (with respect to the metric g) of a smooth function $f : M \to \mathbf{R}$ is the vector field $\operatorname{grad} f$ on M defined by

$$\operatorname{grad} f = \mathsf{g}^{-1}(df). \tag{1}$$

(At each $p \in M$, the definition (1) is equivalent to defining $(\operatorname{grad} f)|_p$ to be the unique vector in T_pM such that $g_p((\operatorname{grad} f)|_p \,, u) = \langle df_p \,, u\rangle$ for all $u \in T_pM$.) The same definition applies if f is defined only on some open set U in M; $\operatorname{grad} f$ is then a vector field on U.

(a) Suppose that $f : M \to \mathbf{R}$ is smooth and that Z is a "regular level set" of f, i.e. $Z = f^{-1}(c)$ for some regular value c of f. By the Regular Value Theorem, Z is a codimension-1 submanifold of M. Show that $\operatorname{grad} f$ is a nonvanishing normal vector field along Z.

(b) Let (M, g) be $\mathbf{R}^n$ with the standard metric ($g = g_{\text{Euc}} := \sum_{i=1}^n dx^i \otimes dx^i$), let $U \subset \mathbf{R}^n$ be open, and let $f : U \to \mathbf{R}$ be smooth. Show that $\operatorname{grad} f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}$.

(c) Let $r : \mathbf{R}^n \to \mathbf{R}$ denote Euclidean distance to the origin (i.e. $r(x) = \left(\sum_i (x^i)^2\right)^{1/2}$). Compute $\operatorname{grad}\left(\frac{1}{2}r^2\right)$ on $\mathbf{R}^n$, and $\operatorname{grad} r$ on $\mathbf{R}^n \setminus \{0\}$.

4. Let $j : S^n \to \mathbf{R}^{n+1}$ be the inclusion map of the unit n-sphere into $\mathbf{R}^{n+1}$. The *standard metric* on S^n is $g_{\text{std}} := j^* g_{\text{Euc}}$, where g_{Euc} is the standard metric on $\mathbf{R}^{n+1}$. The *standard orientation* of S^n is the induced orientation on S^n as the boundary of the unit ball $D^{n+1} := \{x \in \mathbf{R}^{n+1} \mid \sum_i (x^i)^2 \le 1\}$.

Below, (S^n, g_{std}) is given its standard orientation.

(a) Let $x \in S^n$. Under the canonical identification of $T_x\mathbf{R}^{n+1}$ with $\mathbf{R}^{n+1}$, with what subspace of $\mathbf{R}^{n+1}$ is T_xS^n identified (in terms of x)?

(b) Define the function $r : \mathbf{R}^{n+1} \to \mathbf{R}$ as in Problem 3. Let ω denote the Riemannian volume form of (S^n, g_{std}). Show that

$$\omega = j^*(\iota_{\operatorname{grad} r}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n+1}). \tag{2}$$

(c) For $1 \le i \le n+1$ define $\bar{x}^i = j^* x^i : S^n \to \mathbf{R}$. Show that ω can also be expressed as follows:

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[1] I have never found a font in LaTeX that gives me the script lower-case "g" I've used class for this map. The character I am using in this assignment is {\sf g}, the letter "g" in the sans-serif font.

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1}\, \bar{x}^i \, d\bar{x}^1 \wedge \cdots \wedge \widehat{d\bar{x}^i} \wedge \cdots \wedge d\bar{x}^{n+1} \ . \tag{3}$$

Recall that in such expressions, the "hat"[2] denotes omission:

$$d\bar{x}^1 \wedge \cdots \wedge \widehat{d\bar{x}^i} \wedge \cdots \wedge d\bar{x}^{n+1} := d\bar{x}^1 \wedge \cdots \wedge d\bar{x}^{i-1} \wedge d\bar{x}^{i+1} \wedge \cdots \wedge d\bar{x}^{n+1}.$$

(d) Show that on the "upper hemisphere" $\{p \in S^n \mid x^{n+1}(p) > 0\}$, we can further rewrite (3) as

$$\omega = (-1)^n \frac{d\bar{x}^1 \wedge d\bar{x}^2 \cdots \wedge d\bar{x}^n}{\bar{x}^{n+1}} \ . \tag{4}$$

(e) Recall that the canonical identification of each tangent space of $\mathbf{R}^{n+1}$ with $\mathbf{R}^{n+1}$ gives us an identification of {vector fields on $\mathbf{R}^{n+1}$} with {$\mathbf{R}^{n+1}$-valued functions on $\mathbf{R}^{n+1}$}. This identification gives meaning to the notion of "constant vector field" on $\mathbf{R}^{n+1}$, namely a vector field that corresponds to a constant $\mathbf{R}^{n+1}$-valued function. For each $v \in \mathbf{R}^{n+1}$, let $\tilde{Y}^{(v)}$ be the corresponding "constant" vector field on $\mathbf{R}^{n+1}$, and let $Y^{(v)}$ be the (tangent) vector field on S^n defined by $Y^{(v)}(x) := Y^{(v)}|_x := \pi_x(\tilde{Y}^{(v)}_x)$, where $\pi_x : T_x\mathbf{R}^{n+1} \to T_xS^n$ is orthogonal projection.

For each $v \in \mathbf{R}^{n+1}$, also define a function $f_v : S^n \to \mathbf{R}$ by $f_v(x) = v \cdot x$ (ordinary dot-product).

Show that, for each $v \in \mathbf{R}^{n+1}$,

$$\operatorname{grad} f_v = Y^{(v)}$$

5. Let M be a manifold, let ω, η be differential forms on M, and let X, Y be vector fields on M.

(a) Discover a simple relation between $\iota_X(\iota_Y\omega)$ and $\iota_Y(\iota_X\omega)$.

(b) Let $k = \deg(\omega)$. Show that

$$\iota_X(\omega \wedge \eta) = (\iota_X\omega) \wedge \eta + (-1)^k \omega \wedge \iota_X\eta. \tag{5}$$

Remark. A $\mathbf{Z}_2$-graded algebra is an algebra of the form $A_0 \oplus A_1$ and where, if $\bullet$ is the product on the algebra, $\mathcal{A}_j \bullet \mathcal{A}_k \subset \mathcal{A}_{j+k}$ (and where "$j+k$" is interpreted mod 2). A $\mathbf{Z}$-graded algebra is an algebra of the form $\oplus_{k\in\mathbf{Z}}\mathcal{A}_k$, where each $\mathcal{A}_k$ is a vector space and where, if $\bullet$ is the product on the algebra, $\mathcal{A}_j \bullet \mathcal{A}_k \subset \mathcal{A}_{j+k}$. (If we are given an algebra of the form $\oplus_{k\in\mathbf{N}}\mathcal{A}_k$ or $\oplus_{k\in\mathbf{N}\cup\{0\}}\mathcal{A}_k$, obeying the product relation above, we regard it as a $\mathbf{Z}$-graded algebra in which $A_i = \{0\}$ for $i \le 0$ or $i < 0$, respectively.) Given a $\mathbf{Z}$-graded algebra $\oplus_i A_i$, we can construct an associated $\mathbf{Z}_2$-graded algebra

[2] LaTeX command \widehat{}

$B_0 \oplus B_1$ by setting $B_0 = \oplus_{i \text{ even}} A_i$, $B_1 = \oplus_{i \text{ odd}} A_i$. A linear operator on a $\mathbf{Z}_2$-graded algebra or $\mathbf{Z}$-graded algebra obeying the relation (5), with wedge-product replaced by $\bullet$, is called a *graded derivation*, an *antiderivation*, or a *signed derivation*.

6. Let g, h be (Riemannian metrics) on a manifold M. We say that h is *conformally equivalent* or *conformal* to g if h is a real-valued function times g. This function is called a *conformal factor*. Automatically from this definition, a conformal factor must be smooth and strictly positive, so it is the square of another smooth, strictly positive function. Thus the definition above can be rewritten as: h is conformal to g if

$$h = f^2 g \tag{6}$$

for some smooth, positive function $f : M \to \mathbf{R}$. It is easy to see that "conformal to" is an equivalence relation on the set of metrics on M, so we also say that the metrics g, h are conformally equivalent (to each other), and that the Riemannian manifolds (M, g) are conformally equivalent, when one metric is conformal to the other.

(a) Suppose g, h are conformally equivalent metrics on an oriented n-dimensional Riemannian manifold M, and let $f : M \to \mathbf{R}$ be as in (6). Let ω_g, ω_h be the corresponding Riemannian volume forms. Express ω_h in terms of f and ω_g. (If you do this correctly, you will see one of the reasons we choose to write the conformal factor in the form f^2 rather than just f.)

(b) Suppose $h = c^2 g$ for some constant $c > 0$, a very special case of conformal equivalence, and that M is oriented. Let $\mathrm{Vol}(M, g)$ and $\mathrm{Vol}(M, h)$ denote the volume of the manifold M with respect to the metrics g, h respectively. Express $\mathrm{Vol}(M, h)$ in terms of $\mathrm{Vol}(M, g)$ and c.

Observe (for use in later problems) that the same relationship holds for volumes of a domain $D \subset M$ with regular (=smooth) boundary. (Smoothness of ∂D is not essential here. The theorems we proved for integration of differential forms can be extended to *manifolds with corners*, in which the images of chart-maps are allowed to be open sets in $(\mathbf{R}_+)^n = [0, \infty) \times [0, \infty) \times \cdots \times [0, \infty)$.)

(c) Let $n \geq 1$ and let g_{std} denote the standard metric on the n-sphere S^n (the same metric as in problem 4 above). Let $p_N = (0, 0, \ldots, 0, 1)$, the "north pole" of S^n, and let $\mathrm{ster} : S^n \setminus \{p_N\} \to \mathbf{R}^n$ denote the corresponding stereographic-projection map. We can pull the metric g_{std} back to $\mathbf{R}^n$ by the map ster^{-1}. Explicitly compute $(\mathrm{ster}^{-1})^* g_{\text{std}}$ in terms of g_{Euc} and r, where g_{Euc} is the Euclidean metric on $\mathbf{R}^n$ (not to be confused with the "g_{Euc}" in problem 4, which was the Euclidean metric on $\mathbf{R}^{n+1}$), and the function $r : \mathbf{R}^n \to \mathbf{R}$ is the Euclidean distance to the origin.

If you do this correctly, you will find that $(\mathrm{ster}^{-1})^* g_{\text{std}}$ is conformal to g_{Euc}, that the conformal factor is a function of r, and that this function of r is the same for all n.

7. **Balls and spheres in Euclidean space.** In this problem, $\mathbf{R}^n$ is given its standard Riemannian metric g_{Euc} and orientation. We assume $n \geq 1$.

In this problem, don't forget that *volume* of a subset of an n-dimensional manifold means "n-dimensional volume" (what you're used to calling length if $n = 1$, area if $n = 2$, and volume if $n = 3$). For example, "$\mathrm{Vol}(S^2)$" is what you would have called the surface area of the sphere S^2 in Calculus 2 or 3.

Notation for this problem: (i) "c" always denotes a positive real number. (ii) Let D_c^n and S_c^{n-1} denote, respectively, the closed disk (:= closed ball) and sphere of radius c centered at the origin in $\mathbf{R}^n$. (For $c = 1$, we allow ourselves to omit the subscript, as in problem 4: $D^n = D_1^n, S^{n-1} = S_1^{n-1}$.) Observing that S_c^{n-1} is the boundary of D_c^n in $\mathbf{R}^n$, we give S_c^{n-1} the induced orientation. (iii) Let $j_c : S_c^{n-1} \to \mathbf{R}^n$ denote the inclusion map. (iv) Let $\lambda_c : \mathbf{R}^n \to \mathbf{R}^n$ be the map $x \mapsto cx$. (v) Let $g_c = j_c^* g_{\mathrm{Euc}}$, the induced Riemannian metric on S_c^{n-1}. (When $c = 1$, we also write g_{std} for g_1.) (vi) Let $\omega_{S_c^{n-1}} \in \Omega^{n-1}(S_c^{n-1})$ be the Riemannian volume form of (S_c^{n-1}, g_c) with the given orientation. (vii) Let $\mathrm{Vol}(D_c^n)$, $\mathrm{Vol}(S_c^{n-1})$ denote the volumes of D_c^n and S_c^{n-1} with respect to the metrics g_{Euc} and g_c respectively. (viii) Let $\pi_{\mathrm{rad}} : \mathbf{R}^n \setminus \{0\} \to S^{n-1}$ denote the radial projection map, $x \mapsto x/\|x\|$, where $\|\cdot\|$ is the Euclidean norm. (The real number π will come up in this problem, so we are not simply writing "π" for π_{rad}.) (ix) In equation (7) below, r is the function "distance to the origin" on the domain $\mathbf{R}^n \setminus \{0\}$ (on which r is smooth). Elsewhere in this problem, interpret "r" just as an arbitrary positive number (like c, but more suggestive of "radius") or as a real variable in $(0, \infty)$.

(a) Show that $g_c = c^2 (\lambda_{1/c})^* g_{\mathrm{std}}$.

(b) Show that $\mathrm{Vol}(D_r^n) = r^n \mathrm{Vol}(D^n)$ and that $\mathrm{Vol}(S_r^{n-1}) = r^{n-1} \mathrm{Vol}(S^{n-1})$.

(c) Let $\omega_{\mathbf{R}^n}$ denote the standard volume form on $\mathbf{R}^n$. Show that, on the open subset $\mathbf{R}^n \setminus \{0\}$,

$$\omega_{\mathbf{R}^n} = r^{n-1} dr \wedge \pi_{\mathrm{rad}}^* \omega_{S^{n-1}}. \tag{7}$$

(d) We define $\mathrm{Vol}(D^0) = 1$. (The disk D^0 is a point; we are defining the 0-dimensional volume of this point to be 1.)

In this part you will compute $\mathrm{Vol}(D^n)$ for all $n \geq 0$ explicitly by an "obvious" approach, induction on dimension. There are faster, slicker ways than this method.

(i) Using part (b), show that for $n > 1$,

$$\begin{aligned} \mathrm{Vol}(D^n) &= \int_{-1}^{1} \mathrm{Vol}(D^{n-1}_{\sqrt{1-(x^n)^2}}) dx^n \\ &= \mathrm{Vol}(D^{n-1}) \int_0^\pi \sin^n \theta \, d\theta. \end{aligned} \tag{8}$$

(In case it's hard to read: the object in parentheses on the right-hand side of (8) is the disk of radius $\sqrt{1-(x^n)^2}$ in $\mathbf{R}^{n-1}$.)[3]

(ii) For $n \geq 0$, define

$$I_n = \int_0^\pi \sin^n\theta\, d\theta.$$

(For the case $n = 0$, interpret $\sin^n\theta$ as 1 even when $\theta = 0$ or $\theta = \pi$.) Show that for $n \geq 2$,

$$I_n = \frac{n-1}{n} I_{n-2}\ . \tag{9}$$

(This is a Calculus 2 exercise in integration-by-parts.)

(iii) For positive integers n, we define

$$n!! = n(n-2)(n-4)\ldots \begin{cases} 2, & n \text{ even}, \\ 1, & n \text{ odd}. \end{cases}$$

(So "!!" is like factorial, except that we step down by 2 for the next factor in the product instead of by 1; "$n!!$" does not mean the huge number $(n!)!$. Your professor did not invent this notation of questionable wisdom, but admits to finding it convenient.) We define $0!! = 1$.

Use equation (9) to deduce that for $n \geq 1$,

$$I_n = \frac{(n-1)!!}{n!!} \times \begin{cases} \pi, & n \text{ even}, \\ 2, & n \text{ odd}, \end{cases} \tag{10}$$

and using (10) deduce that

$$I_n\, I_{n-1} = \frac{2\pi}{n}\ . \tag{11}$$

(iv) Using step (i), for $n \geq 2$ we have

$$\mathrm{Vol}(D^n) = I_n \mathrm{Vol}(D^{n-1}) = I_n\, I_{n-1} \mathrm{Vol}(D^{n-2}).$$

Now use equation (11) to derive an explicit formula for $\mathrm{Vol}(D^n)$ and (using part (b)) for $\mathrm{Vol}(D^n_r)$. Your formula should agree with familiar formulas for the cases $n = 1, 2, 3$. It should also yield the amusing formula

[3] For the case $n = 3$, the angle θ in the second integral above is the angle between the positive z-axis and the line-segment from the origin to a point on the sphere. Physicists usually call this angle θ (or at least they used to, when your professor was a student), but your Calculus 3 textbook probably called it ϕ. Outside of calculus classes, it's more common to use the physics convention for which spherical coordinate is called θ and which is called ϕ (the opposite of the calculus-textbook convention). One case in which the physics convention is completely standard is the notation for spherical harmonics (see Wikipedia's article on this topic, for example).

$$\sum_{m=0}^{\infty} \mathrm{Vol}(D^{2m}) = e^{\pi}. \qquad (12)$$

(As far as your professor knows, equation (12) is devoid of any deep meaning and is entirely useless.)

(e) Using part (c) and a homework problem from last semester, show that

$$\mathrm{Vol}(D^n_r) = \int_0^r \mathrm{Vol}(S^{n-1}_t)\ dt \qquad (13)$$

$$= \frac{r^n}{n}\mathrm{Vol}(S^{n-1}). \qquad (14)$$

Observe that (13) implies an equality you might have guessed,

$$\frac{d}{dr}\mathrm{Vol}(D^n_r) = \mathrm{Vol}(S^{n-1}_r), \qquad (15)$$

and thereby get a formula for $\mathrm{Vol}(S^n)$ for all $n \geq 1$.

(f) Using (14) and part (d)(iv), obtain an explicit formula for $\mathrm{Vol}(S^n)$. (This should agree with familiar formulas for $n = 1, 2$.)

Remark. An alternative inductive approach to finding the volumes of Euclidean spheres, without first finding the volumes of Euclidean balls, is to use problem 4(d) and equations (7) and (14) to get a recurrence relation between $\mathrm{Vol}(S^n)$ and $\mathrm{Vol}(S^{n-1})$. Similar trigonometric integrals are involved.

8. **Measure-zero sets in manifolds**. *Doing* this long problem is **optional**, but you should *read* it whether or not you do any work on the problem. The concepts and results are worth knowing. You'll be allowed to apply them (in this course) without doing this problem.

"Set of measure zero" in $\mathbf{R}^n$, or in any n-dimensional manifold, is a concept that does not require any knowledge of measure theory or any graduate-level real analysis. It's definable without having to define what "measure" means for any set that's *not* of measure zero.

Definition 1: A *cube* of side λ in $\mathbf{R}^n$ is the Cartesian product of n closed intervals of length λ. Equivalently, a cube of side λ in $\mathbf{R}^n$ is a closed ball of radius $\lambda/2$ with respect to the norm $\|\cdot\|_\infty$ on $\mathbf{R}^n$ ($\|(x^1,\ldots,x^n)\|_\infty = \max_i |x_i|$).

Definition 2: A set $Z \subset \mathbf{R}^n$ has *measure zero* (as a subset of $\mathbf{R}^n$) if, for all $\epsilon > 0$, Z can be covered by a countable collection of cubes the sum of whose volumes is less than ϵ.[4]

Observe that, trivially, every subset of a measure-zero set in $\mathbf{R}^n$ has measure zero (as a subset of $\mathbf{R}^n$).

(a) Show that, as a subset of $\mathbf{R}^2$, the set $[0,1] \times \{0\}$ has measure zero.

(b) Show that the countable union of measure-zero subsets of $\mathbf{R}^n$ has measure zero. Using this and part (a) (suitably generalized), show that the x-axis $\{(x,0) \mid x \in \mathbf{R}\}$ has measure zero in $\mathbf{R}^2$.

(c) Generalize part (b): show that if $1 \leq k < n$, then $\mathbf{R}^k \times \{0 \in \mathbf{R}^{n-k}\}$ has measure zero in $\mathbf{R}^n$.

(d) Let $U \subset \mathbf{R}^n$ be open, $f : U \to \mathbf{R}^n$ a C^1 map, $K \subset U$ compact. Show that there exists $c > 0$ if $C \subset K$ is a cube of side λ, then $f(C)$ is contained in a cube of side $c\lambda$. (The constant c may depend on n, f, and K.)

(e) Let $U \subset \mathbf{R}^n$ be open, $f : U \to \mathbf{R}^n$ a continuously differentiable map. Use parts (c) and (b) to show that if $Z \subset \mathbf{R}^n$ has measure zero, then $f(Z)$ has measure zero. (You will also need the fact that every open subset of $\mathbf{R}^n$ is σ-compact. This follows trivially from Theorem 0.11 in the "Sufficient conditions for paracompactness" notes, but can be proven more directly.)

(f) Let $(U,\phi), (V,\psi)$ be charts of an n-dimensional manifold M, and let $Z \subset U \bigcap V$. Show that if $\phi(Z)$ has measure zero in $\mathbf{R}^n$, then so does $\psi(Z)$.

Definition 3. Let M be an n-dimensional manifold. We say that a subset $Z \subset M$ *has measure zero* if for all charts (U,ϕ) of M, the set $\phi(Z \bigcap U)$ has measure zero in $\mathbf{R}^n$. (Note that "all charts" means "all charts in the *maximal* atlas of M".)

(g) Let M be an n-dimensional manifold, $Z \subset M$. Using part (f), show that a sufficient condition for Z to have measure zero is that for *some* atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$, not necessarily maximal, $\phi_\alpha(Z \bigcap U_\alpha)$ has measure zero in $\mathbf{R}^n$ for all $\alpha \in A$.

(h) Let M be an n-dimensional manifold, $Z \subset M$ a manifold of positive codimension. Show that Z has measure zero in M.

(i) Let M be an n-dimensional oriented manifold, $Z \subset M$ a closed submanifold of positive codimension. For simplicity, assume that M is compact (this assumption is not necessary). Since Z is closed, $M \setminus Z$ is an open set in M, hence an oriented manifold. Let ω be a compactly supported n-form on M. Show that

$$\int_M \omega = \int_{M \setminus Z} \omega. \tag{16}$$

[4] There are two conventions for what "countable" means. The one your professor uses is that a set is countable if it can be put into one-to-one correspondence with a subset of the natural numbers. Thus a countable set is either finite or countably infinite.

(To do this problem, part of what you will need to do is to figure out how to define the right-hand side of (16), since $\omega|_{M\setminus Z}$ need not have compact support.)

(j) Hypotheses as in part (i), but now take the closed set Z to be a finite union of positive-codimension submanifolds of M. Show that (16) still holds.

Remark. What you are showing in the last two parts of this problem are special cases of the principle, "For purposes of integration, sets of measure zero do not matter." This is true with far fewer hypotheses than were given above, but more work and more definitions are needed to show this.

9. **Application of Problem 8: computing integrals in practice.** Partitions of unity are an indispensable tool in defining, and proving theorems about, integrals of differential forms. However, for purposes of *computing* most integrals, partitions of unity are impractical; their formulas are too complicated for the relevant iterated integrals to be calculated explicitly. Instead, to integrate an n-form ω over an oriented manifold M, what we usually do is to cover *most* of M—everything but a closed set of measure zero, usually a finite union of submanifolds of positive codimension—with a finite number of disjoint charts whose closures cover all of M. Often all we need is *one* chart.

Read the examples below; the problem-parts are after the examples.

Example 1: $M = S^n \subset \mathbf{R}^{n+1}$, with the standard orientation. We can write M as the union of the open upper hemisphere H_+, the open lower hemisphere H_-, and the equator. The equator is a codimension-one submanifold, hence has measure zero. On each hemisphere we can use the chart-map given by the projection $(x^1, \dots, x^n, x^{n+1}) \mapsto (x^1, \dots, x^n)$. For any $\omega \in \Omega^n(S^n)$, we have $\int_M \omega = \int_{H_+} \omega + \int_{H_-} \omega$. We can compute the integrals on the right-hand side by pulling them back to the open unit disk $(D^n)^\circ \subset \mathbf{R}^n$ using the inverses of the chart-maps, being careful with orientations. (The given chart on H_- is positively oriented, but the given chart on H_- is negatively oriented. If f is the inverse of the chart-map on H_-, we have $\int_{H_-} \omega = -\int_{(D^n)^\circ} f^*\omega$.)

Example 2: Again $M = S^n \subset \mathbf{R}^{n+1}$, with the standard orientation. As in problem 6c, let $p_N \in M$ be the "north pole" as in problem 6c and let $\text{ster} : U := M \setminus \{p_0\} \to \mathbf{R}^n$ be the corresponding stereographic projection map. Then for any $\omega \in \Omega^n(S^n)$, we have $\int_M \omega = \int_U \omega$, which we can evaluate by pulling back to $\mathbf{R}^n$ by ster^{-1}. Unless ω vanishes on a neighborhood of the north pole, $(\text{ster}^{-1})^*\omega$ will not have compact support, so the integral over $\mathbf{R}^n$ will be an *improper* integral in the sense of Calculus 2, but it is *guaranteed to converge* because $\int_M \omega$ exists.

Example 3: $M = S^2 \subset \mathbf{R}^3$, with the standard orientation. We can use spherical coordinates (see footnote 3) to parametrize the sphere: define $f : (0, \pi) \times (0, 2\pi) \to S^2$ by $f(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. With this domain the map f is one-to-one, and its image is $S^2 \setminus Z$, where Z is a closed semicircle connecting the north and south poles, lying in the half-plane $\{(x, y, z) \mid y = 0, x \geq 0\}$. The pair $(S^2 \setminus Z, f^{-1})$ is a positively-oriented chart of S^2. (Students: check that "positively oriented" is correct here.) The set Z is the union of two zero-dimensional submanifolds and one one-dimensional submanifold, hence has measure zero. Hence for any $\omega \in \Omega^2(S^2)$,

$$\int_{S^2} \omega = \int_{(0,\pi)\times(0,2\pi)} f^*\omega.$$

The open rectangle $(0, \pi) \times (0, 2\pi)$ is not compact, of course. But the map f can be extended to a smooth map $\tilde{f} : \mathbf{R}^2 \to S^2$ by removing the restrictions on θ and φ. Then $\tilde{f}^*\omega$ can be integrated over the compact rectangle $[0, \pi] \times [0, 2\pi]$. Since the boundary of this rectangle has measure zero in $\mathbf{R}^2$ (it is a finite union of zero-dimensional and one-dimensional submanifolds of $\mathbf{R}^2$), and the restriction of $\tilde{f}$ to the open rectangle is f, we have

$$\begin{aligned} \int_{(0,\pi)\times(0,2\pi)} f^*\omega &= \int_{[0,\pi]\times[0,2\pi]} \tilde{f}^*\omega \\ &= \int_0^{\pi} \left[\int_0^{2\pi} \{\text{some smooth function of } (\theta, \varphi)\} d\varphi \right] d\theta. \end{aligned}$$

(The function of (θ, φ) in the integrand will depend on ω, of course; the purpose of the second line above is just to remind you that the way we would usually compute the double integral on the previous line is to turn it into an interated integral.)

Example 4. Recall from last semester's homework that the $(2n)$-dimensional manifold $\mathbf{C}P^n$, complex projective space, has a "standard atlas" with $n + 1$ charts (U_i, ϕ_i), where $U_i = \{[z^0, z^1, \ldots, z^n] \mid z^i \neq 0\}$. **Fact**: $\mathbf{C}P^n$ is orientable. (You may assume this, or try to show it if you wish.) The standard orientation is the one for which ϕ_0 (or any ϕ_i) is an orientation-preserving map from its domain to $\mathbf{R}^{2n}$. The complement of any of the sets U_i is a submanifold of $\mathbf{C}P^n$ diffeomorphic to $\mathbf{C}P^{n-1}$ (why?), hence has measure zero in $\mathbf{C}P^n$. Thus, for any $\omega \in \Omega^{2n}(\mathbf{C}P^n)$,

$$\int_{\mathbf{C}P^n} \omega = \int_{U_0} \omega = \int_{\mathbf{R}^{2n}} ({\phi_0}^{-1})^*\omega.$$

(a) Compute $\text{Vol}(S^2)$ (with respect to the standard metric) using Example 1, 2, or 3 (your choice) and the Riemannian volume form on S^2 computed in problem 4. **Optional**: Compute the volume using more than one of these examples and check that you get the same answer.

(b) (**This part is optional.**) **Fact you may assume:** There is a differential form $\omega \in \Omega^4(\mathbf{C}P^2)$ for which

$$({\phi_0}^{-1})^*\omega = \frac{dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2}{(1 + |z^1|^2 + |z^2|^2)^4} \tag{17}$$

where $z^j = x^j + iy^j$, $j = 1, 2$. (The statement above is true with the last exponent in the denominator replaced by any $m \geq 4$, but there is something special about the exponent 4: the formulas for $({\phi_1}^{-1})^*\omega$ and $({\phi_2}^{-1})^*\omega$ are also (17), modulo the names of the coordinates. There's a good reason for this, but the explanation requires a digression into complex-valued coordinates and differential forms. For $\mathbf{C}P^n$ the magic exponent is $2n$.) Compute $\int_{\mathbf{C}P^2} \omega$. You will need $\mathrm{Vol}(S^3)$ from problem 7f to finish this computation.