Differential Geometry 2—MAT 4930 —Spring 2015
Assignment 2

1. Let M, N be manifolds and $F : M \to N$ a smooth map. Vector fields $\tilde{X}$ on M, X on N are said to be *F-related* if $F_{*p}\tilde{X}_p = X_{F(p)}$ for all $p \in M$. We also sometimes say that $\tilde{X}$ *projects* to X if this relation holds, but that terminology can be misleading if F is not surjective. (F could even be the inclusion map of a submanifold, and $\tilde{X}$ an extension of X to N.) Note that a necessary condition for a vector field $\tilde{X}$ to be "projectable by F"—i.e. F-related to some vector field on N—is that for all $q \in N$ and all $p_1, p_2 \in F^{-1}(\{q\})$, we must have $F_{*p_1}\tilde{X}_{p_1} = F_{*p_2}\tilde{X}_{p_2}$. Most vector fields on M will not meet this consistency condition if F is not injective.

Suppose that $M, N, F, \tilde{X}$, and X are as above, with $\tilde{X}$ F-related to X. Let $\tilde{\Phi}$ and Φ be the flows of $\tilde{X}$ and X respectively, defined on their maximal domains.

(a) Show that if $(p, t) \in \text{domain}(\tilde{\Phi})$ then $(F(p), t) \in \text{domain}(\Phi)$ (so that if $\tilde{\Phi}_t(p)$ is defined, then so is $\Phi_t(F(p))$), and that $F \circ \tilde{\Phi}_t = \Phi_t \circ F$ on domain$(\tilde{\Phi})$.

(b) Show that if $\tilde{Y}$ is another vector field on M, and is F-related to a vector field Y on N, then $[\tilde{X}, \tilde{Y}]$ is F-related to $[X, Y]$. (Use the fact that $[X, Y] = \mathcal{L}_X Y$.)

We often write the fact proven in (b) as "$F_*[\tilde{X}, \tilde{Y}] = [F_*\tilde{X}, F_*\tilde{Y}]$", with the understanding that this applies only if $\tilde{X}, \tilde{Y}$ are projectable by F.

2. Let X be a vector field on the manifold M, with flow Φ. Let μ be a tensor field on M. Recall that the Lie derivative of μ by X at the point p is defined by

$$(\mathcal{L}_X\mu)|_p = \frac{d}{dt}\left((\Phi_t^*\mu)|_p\right)\Big|_{t=0}.$$

It is natural to ask: what if we evaluate the t-derivative at general t?

Show that if (p, t_0) is in the domain of the flow, then

$$\frac{d}{dt}\left((\Phi_t^*\mu)|_p\right)\Big|_{t=t_0} = \left(\Phi_{t_0}^*(\mathcal{L}_X\mu)\right)\big|_p. \tag{1}$$

Remark. We allow ourselves to rewrite (1) more briefly as

$$\frac{d}{dt}\Phi_t^*\mu = \Phi_t^*(\mathcal{L}_X\mu) \tag{2}$$

with understanding that the equation is interpreted pointwise; for each $p \in M$ we are differentiating the curve $t \mapsto (\Phi_t^*\mu)_p$ in a *fixed, finite-dimensional* vector space, the fiber at p of the appropriate tensor bundle. For each p, there is guaranteed to be an open interval containing 0 on which this curve is defined. If we attempt, instead, to interpret $t \mapsto \Phi_t^*\mu$ as a curve in the infinite-dimensional vector space $\Gamma(E)$, the space of sections of the tensor bundle in which μ takes its values, we run into two problems:

(i) if the (maximal) domain of Φ is not all of $M \times \mathbf{R}$, then there will be *no* $\epsilon > 0$ for which "$\Phi_t^* \mu$" is a section of E (a tensor field defined on all of M) for all $t \in (-\epsilon, \epsilon)$; and (ii) even if Φ is defined on $M \times \mathbf{R}$, we would have to choose a topology on $\Gamma(E)$ in order for the difference-quotient limit to be defined (and furthermore, the limit might exist for some topologies and not for others, and for some μ but not for others).

(b) Show that for all (p, t) in the domain of the flow Φ of X, we have

$$(\Phi_t^* X)_p = X_p. \tag{3}$$

(One way to get this is to use part (a). However, (3) is also equivalent to $X_{\Phi_t(p)} = \Phi_{t*p} X_p$, which can be shown directly, without using part (a).) We usually write (3) more briefly as "$\Phi_t^* X = X$" again with the understanding that this is to be interpreted pointwise in M.

3. Let M be a manifold, X a vector field on M, Φ the flow of X. Prove that fixed-points of the flow—i.e. those points $p \in M$ for which $\Phi_t(p) = p\ \forall t$—are exactly those p at which $X_p = 0$.

4. Notation as in problem 3. An *integral curve* of X is a "Φ-orbit", i.e. a set of the form $\{\Phi_t(p)\}$ with p fixed and t varying over an open interval containing 0. Prove that multiplying X by a nonzero function only reparametrizes the integral curves; it does not change the underlying point-sets. More precisely, if $Y = fX$ for some nonzero function f, prove that every integral curve of Y is an integral curve of X and vice-versa.

5. Let M be a manifold, X a vector field on M, and suppose that $X_p \neq 0$ at some $p \in M$. Prove that there are local coordinates $\{x^i\}$ on some open neighborhood U of p such that $X = \partial/\partial x^1$ on U. (Hint: use the flow of X, and turn time into a coordinate.)

Note: "there are local coordinates $\{x^i\}$ on some open neighborhood U of p" is just another way of saying "there is a chart (U, ϕ) with $p \in U$"; it is understood that the $\{x^i\}$ are the standard coordinate functions on $\phi(U) \subset \mathbf{R}^n$, $(n = \dim(M))$ pulled back to U.

6. This problem begins an important generalization of problem 5.

The integral curves of a nonzero vector field on a manifold M are one-dimensional submanifolds of M. Given *two* vector fields X and Y, one can ask whether their integral curves "hang together" to produce two-dimensional submanifolds (at least locally), any two points of which can be connected to each other by moving along a piecewise-smooth curve each of whose smooth segments is a portion of an integral curve of X or Y. This problem gives a sufficient condition on X and Y for them to generate a two-dimensional submanifold locally in this way. The condition is also necessary in a certain sense (see part (c)). A more general sufficient condition is given in part (d). (Note: Each part of this problem after (a) depends on part (a).)

Let M be a manifold, X and Y vector fields on M. , with flows Φ and Ψ respectively. We say *the flows of X and Y commute* if for any $p \in M$ and any open intervals I, J containing 0 such that if $\Phi_t \circ \Psi_s(p)$ and $\Psi_s \circ \Phi_t(p)$ are defined for all $(t, s) \in I \times J$, then $\Phi_t \circ \Psi_s(p) = \Psi_s \circ \Phi_t(p)$ for all $(t, s) \in I \times J$.

(a) Prove that $[X, Y] \equiv 0$ iff the flows of X and Y commute. (One direction of the "iff" is much easier than the other. The hard direction is what's needed below.)

(b) Suppose that X and Y are linearly independent at p, and that $[X, Y] \equiv 0$. Prove that there are local coordinates $\{x^i\}$ on some open neighborhood U of p such that $X = \partial/\partial x^1$ and $Y = \partial/\partial x^2$ on U. (Hint: use the flows.)

(c) Hypotheses as in (b). Prove that there exists a two-dimensional submanifold L of M, containing p, such that at each point q of L the tangent space T_qL is spanned by X_q and Y_q. Conversely (more or less), show that given any two-dimensional submanifold L of M, and any $p \in L$, there exist vector fields X, Y defined on an open neighborhood U of p in M that are linearly independent at every point of U and satisfy $[X, Y] \equiv 0$ on U. (The reason for the "more or less" is that your X and Y are required to be linearly independent not just at p, but throughout U. However, you will probably find that the same proof you'd have used to show linear independence at p works at every point of your U.)

(d) Suppose that on some open set V, the vectors X_q and Y_q are linearly independent at each $q \in V$ and that $[X, Y]_q$ lies in the span of X_q and Y_q. Thus there are unique functions $f, g : V \to \mathbf{R}$ such that $[X, Y] = fX + gY$ for some unique functions f and g.

(i) Show that f and g are smooth.

(ii) (You may assume part (i) to do this part.) Prove for each $p \in V$ there exist a neighborhood U of p and locally-defined vector fields $\tilde{X}, \tilde{Y}$ with the same span as X and Y at each point of U, satisfying $[\tilde{X}, \tilde{Y}] \equiv 0$. (Hence, from part (c) these conditions on X and Y can replace the less general conditions in part (b).)