Differential Geometry 2—MAT 4930 —Spring 2015
Assignment 3

1. Recall (especially if you've never learned it) the *polar decomposition* of an invertible matrix: any invertible matrix A can be written uniquely as a product RS, where R is an orthogonal matrix and S is a symmetric positive-definite matrix. ("Recall" also that a symmetric $n \times n$ real matrix S is called *positive-definite* if $v \cdot Sv > 0$ for all nonzero $v \in \mathbf{R}^n$, and that this condition is equivalent to positivity of all the eigenvalues of S.) It can be shown that R and S depend smoothly on A. It can also be shown that a 2×2 symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive-definite if and only if $a > 0$ and $\det(A) > 0$. Below, you may assume all of the facts just stated.

(a) Let $\mathcal{S} = \{2 \times 2 \text{ symmetric real matrices}\}$, a 3-dimensional vector space. Let $\mathcal{S}_1 = \{S \in \mathcal{S} \mid \det(S) = 1\}$. Show that $\mathcal{S}_1$ is a submanifold of $\mathcal{S}$ diffeomorphic to $\mathbf{R}^2$.

(b) Use the polar decomposition and part (a) to show that $SL(2, \mathbf{R})$ is diffeomorphic to $SO(2) \times \mathbf{R}^2$, hence to $S^1 \times \mathbf{R}^2$.[1] This shows, in particular, that $SL(2, \mathbf{R})$ is connected.

(c) Recall from class that the Lie algebra of $SL(2, \mathbf{R})$ is $\mathfrak{sl}(2, \mathbf{R}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbf{R} \right\}$. In class we saw that if $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, then $X^2 = \Delta I$, where $\Delta = a^2 + bc$. Show that

$$\exp(X) = \begin{cases} (\cosh \sqrt{\Delta}) I + \frac{\sinh \sqrt{\Delta}}{\sqrt{\Delta}} X & \text{if } \Delta > 0, \\ I + X & \text{if } \Delta = 0, \\ (\cos \sqrt{|\Delta|}) I + \frac{\sin \sqrt{|\Delta|}}{\sqrt{|\Delta|}} X & \text{if } \Delta < 0. \end{cases} \tag{1}$$

(d) Deduce from part (c) that for all $X \in \mathfrak{sl}(2, \mathbf{R})$, $\mathrm{tr}(\exp(X)) \geq -2$.

(e) Find $A \in SL(2, \mathbf{R})$ with $\mathrm{tr}(A) < -2$, and hence show that $SL(2, \mathbf{R})$ is a connected Lie group for which the exponential map is not surjective.

2. Another Lie group for which it is easy to compute the exponential map explicitly is $SO(3)$.

(a) As with other matrix groups, we identify $\mathfrak{so}(n)$, the Lie algebra of $SO(n)$, with $T_I(SO(n)) \subset T_I M_n(\mathbf{R}) \cong_{\text{canon.}} M_n(\mathbf{R})$, and thereby identify $\mathfrak{so}(n)$ with a subspace of

[1] *Note*: "Diffeomorphic" is all you're asked to show, not "isomorphic as Lie groups". The Lie groups $SL(2, \mathbf{R})$ and $SO(2) \times \mathbf{R}^2$ are *not isomorphic*. If they were, they'd have isomorphic Lie algebras. But the Lie algebra of $SO(2) \times \mathbf{R}^2$ is abelian, while the Lie algebra of $SL(2, \mathbf{R})$ is not.

$M_n(\mathbf{R})$. Show that $T_I(SO(n)) = \ker(DF|_I)$, where $F : M_n(\mathbf{R}) \to \{$symmetric $n \times n$ real matrices$\}$ is defined by $F(A) = A^t A$ (see last semester's Assignment 4 non-book problem 1). Then use this fact to show that $\mathfrak{so}(n)$ is the space of $n \times n$ antisymmetric matrices.

(b) Using the same general procedure as in problem 1(c), compute $\exp : \mathfrak{so}(3) \to SO(3)$. (Note: for $X \in \mathfrak{so}(3)$ you won't find a simple formula relating X^2 to I, but you'll find one relating X^3 to X.) From your explicit formula for this exponential map[2], deduce that all one-parameter subgroups $\gamma : \mathbf{R} \to SO(3)$ are periodic.

3. Let G be a Lie group. A (real-valued) differential form ω on a Lie group G is called *left-invariant* if $L_g^*\omega = \omega$ for all $g \in G$. Just as for a vector field, left-invariance of a "set-theoretic" differential form on G automatically implies smoothness.

(a) Let $k \geq 0$ and let $\xi \in \bigwedge^k T_e^* G$. Show that there exists a unique k-form $\omega \in \Omega^k(G)$ such that $\omega_e = \xi$.

(b) Characterize, in simpler terms, what a left-invariant 0-form on G is.

(c) Let $k \geq 1$, let ω be a left-invariant k-form on G, and let $X_1, \dots, X_k$ be left-invariant vector fields on G. Show that the function $\omega(X_1, \dots, X_k) : G \to \mathbf{R}$ is constant.

(d) Show that if a differential form ω on G is left-invariant, then so is $d\omega$.

(e) Let θ be a left-invariant 1-form on G and let X, Y be left-invariant vector fields. Show that $d\theta(X, Y) = -\langle \theta, [X, Y] \rangle$.

(f) Let V be a finite-dimensional vector space, and for $k \geq 0$ let $\Omega^k(G; V)$ denote the space of V-valued differential forms on G. (Brief review: we define $\Omega^0(G; V)$ to be the space of smooth functions $G \to V$; for $k \geq 1$ an element of $\Omega^k(G; V)$ is a section of the vector bundle whose fiber at $g \in G$ is the space of k-linear antisymmetric maps $T_g G \times T_g G \times \dots T_g G \to V$, a space that we can canonically identify with $(\bigwedge^k T_g^* G) \otimes V$. For $k \geq 1, \omega \in \Omega^k(G; V)$, and vector fields $X_1, \dots, X_k$ on G, $\omega(X_1, \dots, X_k)$ is a smooth function $G \to V$.) The definition of "left-invariant" is the same for V-valued differential forms ω as for real-valued differential forms: $L_g^*\omega = \omega$ for all $g \in G$. Show that parts (a)–(e) of this problem generalize to V-valued differential forms.

4. (**Lie-algebra-valued differential forms.**) (Note: Do problem 3(f) first.) Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. For purposes of this problem, it is more convenient to identify the set $\mathfrak{g}$ with $T_e G$ than with the set of left-invariant vector

[2] This formula, which anyone who has ever bothered to compute the exponential map $\mathfrak{so}(3) \to SO(3)$ has figured out for him/herself in a matter of minutes, is glorified in the applied-math literature with a name, "Rodrigues' formula". In fairness, though, whether or not Rodrigues, whoever he or she was, deserves to have his or her name attached to this formula, there's an advantage to having the name: it's quicker to say "By Rodrigues' formula, ..." than to say "By the standard formula for the exponential map from $\mathfrak{so}(3) \to SO(3)$, ..."

fields. Below, "$\mathfrak{g}$-valued differential form" means "T_eG-valued differential form."

There is a combination "wedge-bracket" operation $\Omega^k(G;\mathfrak{g})\times\Omega^l(G;\mathfrak{g}) \to \Omega^{k+l}(G;\mathfrak{g})$ that can be defined as follows: for $\omega \in \Omega^k(G;\mathfrak{g}), \eta \in \Omega^l(G;\mathfrak{g})$, and vector fields $X_1,\ldots,X_{k+l}$ on G

$$[\omega,\eta](X_1,\ldots,X_{k+l}) = \frac{1}{k!l!}\sum_{\pi\in S_{k+l}} \operatorname{sgn}(\pi)[\omega(X_{\pi(1)},\ldots,X_{\pi(k)})\ ,\ \eta(X_{\pi(k+1)},\ldots,X_{\pi(k+l)})].$$

In particular, for $\omega,\eta \in \Omega^1(G;\mathfrak{g})$, $\mu \in \Omega^2(G;\mathfrak{g})$, and vector fields X, Y, Z on G,

$$\begin{aligned} [\omega,\eta](X,Y) &= [\omega(X),\eta(Y)] - [\omega(Y),\eta(X)] \\ \text{and} \qquad [\omega,\mu](X,Y,Z) &= [\omega(X),\mu(Y,Z)] + [\omega(Y),\mu(Z,X)] + [\omega(Z),\mu(X,Y)]. \end{aligned}$$

Note that for $\omega \in \Omega^1(G;\mathfrak{g})$, the 2-form $[\omega,\omega]$ is nonzero in general:

$$[\omega,\omega](X,Y) = 2[\omega(X),\omega(Y)].$$

The group G has a canonical $\mathfrak{g}$-valued 1-form ω called the *Maurer-Cartan form*, defined by

$$\omega_g = L_{g^{-1}*g} \quad . \tag{2}$$

Equation (2) may look initially as if somebody forgot to put something to the right of $L_{g^{-1}*g}$, but there's no typo here: $L_{g^{-1}*g}$ is a linear map $T_gG \to T_eG = \mathfrak{g}$. Such a linear map is exactly what the value at g of a $\mathfrak{g}$-valued 1-form is supposed to be.

Below, ω denotes the Maurer-Cartan form on G.

(a) Show that ω is left-invariant, and can be characterized as the unique left-invariant $\mathfrak{g}$-valued 1-form on G whose value at e is the identity map $T_eG \to T_eG$. (This is the sense in which the Maurer-Cartan form is canonical.)

(b) Show that ω satisfies the *structural equation*

$$d\omega + \frac{1}{2}[\omega,\omega] = 0.$$

(c) Show that for any $\mathfrak{g}$-valued 1-form η, we have

$$[\eta,[\eta,\eta]] = 0, \tag{3}$$

and that for $\eta = \omega$, equation (3) is equivalent to the Jacobi identity for the Lie algebra $\mathfrak{g}$.