Differential Geometry—MTG 6257—Spring 2000
Problem Set 1

Notation for all the problems below. Unless otherwise specified, (M, g) is a Riemannian manifold of dimension n. Summation notation for repeated indices is used.

1. Let $\{\theta^i\}_1^n$ be a locally-defined orthonormal basis of 1-forms. Prove that there is a *unique* antisymmetric $n \times n$ matrix B of locally-defined 1-forms such that $d\theta^i + B^i{}_j \wedge \theta^j = 0$, $i = 1, \ldots, n$.

2. Let $\tilde{g}$ be a metric on M conformally related to g; thus $\tilde{g} = e^{2f}g$ for some function $f : M \to \mathbf{R}$. Let $\nabla, \tilde{\nabla}$ be the Levi-Civita connections on TM induced by $g, \tilde{g}$ respectively. Prove that for all vector fields X, Y on M,

$$\tilde{\nabla}_X Y = \nabla_X Y + X(f)Y + Y(f)X - g(X, Y)\mathrm{grad}_g(f),$$

where grad_g denotes gradient taken with respect to g.

3. (Notation as in problem 2.) Let $\{e_i\}$ be a local g-orthonormal basis of TM, and let $\tilde{e}_i = e^{-f}e_i$. Let $\{\theta^i\}, \{\tilde{\theta}^i\}$ be the local bases of T^*M dual to $\{e_i\}, \{\tilde{e}_i\}$ respectively, and let $\Theta, \tilde{\Theta}$ be the (matrix-valued) connection forms relative to $\{e_i\}, \{\tilde{e}_i\}$ respectively.

(a) Show that $\{\tilde{e}_i\}$ is $\tilde{g}$-orthonormal.

(b) Show that $\tilde{\theta}^i = e^f\theta^i$, $i = 1, \ldots, n$.

(c) Show that $\tilde{\Theta}^i_j = \Theta^i{}_j - \iota_{\mathrm{grad}_g f}(\theta^i \wedge \theta^j)$.

4. (a) Let $\mathbf{R}^2_+ = \{(x, y) \in \mathbf{R}^2 \mid y > 0\}$, also called the *upper half-plane*. Let g_1 be the Riemannian metric $(dx^2 + dy^2)/y^2$ (recall that dx^2 means $dx \otimes dx$). Compute the Gauss curvature $K(x, y)$ for the Riemannian manifold $(\mathbf{R}^2_+, g_1)$.

(b) Let D^2 be the open unit disk in $\mathbf{R}^2$, i.e. $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$. Let g_2 be the Riemannian metric

$$\frac{4}{(1 - r^2)^2}(dx^2 + dy^2),$$

where $r^2 = x^2 + y^2$. Compute the Gauss curvature $K(x, y)$ for the Riemannian manifold (D^2, g_2).

(c) Find an isometry $F : (\mathbf{R}^2_+, g_1) \to (D^2, g_2)$ (i.e. a diffeomorphism for which $F^*g_2 = g_1$).

(d) Let $c > 0$. Find a metrics on (i) $\mathbf{R}^2_+$, and (ii) D^2, each of which has constant Gauss curvature $-c^2$.

5. Let $\gamma : [a, b] \to M$ be a smoothly parametrized curve for which γ' is nowhere zero. (a) Prove that there exists a smooth reparametrization of γ, say $\tilde{\gamma}$, with the same orientation as γ and for which $\|\tilde{\gamma}'\| \equiv 1$. (This is called a *unit-speed parametrization* or an *arclength parametrization*.) (b) Prove that if $\tilde{g}_1$ and $\tilde{g}_2$ are two such reparametrizations, then their parameters differ by a constant: $\tilde{\gamma}_2(t) = \tilde{\gamma}_1(t + c)$ for some constant c.

6. Let $(Z, \tilde{g})$ be a Riemannian manifold, $M \subset Z$ a submanifold, $\iota : M \to Z$ the inclusion map, and $g = \iota^* \tilde{g}$ (the Riemannian metric on M induced from $(Z, \tilde{g})$—the restriction of $\tilde{g}$ to vectors tangent to M). For $p \in M$, let $\pi : T_{\iota(p)}Z \to T_pM$ be the orthogonal projection defined by $\tilde{g}$. Consider the two connections $\nabla^{(1)}, \nabla^{(2)}$ on (M, g) defined below.

(i) $\nabla^{(1)} =$ the Levi-Civita connection of (M, g).

(ii) $\nabla^{(2)} =$ the operator defined as follows. Let $p \in M$ and let X, Y be vector fields defined on a neighborhood U^M of p in M. Let $\tilde{X}, \tilde{Y}$ be extensions of $\iota_* X, \iota_* Y$ to a neighborhood U^Z of $\iota(p)$ in Z. Define

$$(\nabla^{(2)}_X Y)|_p = \pi(\nabla^Z_{\tilde{X}} \tilde{Y}|_{\iota(p)})$$

where ∇^Z is the Levi-Civita connection of the metric $\tilde{g}$ on TZ. Show that $\nabla^{(2)}_X Y$ is well-defined (i.e. is independent of the choices of extensions) and that $\nabla^{(2)} = \nabla^{(1)}$.

7. Let U be an open neighborhood of the origin in $\mathbf{R}^2$, $f : U \to \mathbf{R}$ a smooth function, and let $M \subset \mathbf{R}^3$ be the graph of f. Assume that $f(0,0) = 0$ and that $D_{(0,0)} f = 0$, so that M passes through the origin and has horizontal tangent plane there. Give $\mathbf{R}^3$ the standard metric, let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the usual unit vectors in $\mathbf{R}^3$, and let $\hat{h}$ be the second fundamental form of M with respect to the unit normal vector field whose value at the origin is $\mathbf{k}$. Let H be the Hessian of f at (0,0) (the matrix of second partials); since f and its first partials vanish at (0,0), Taylor's Theorem implies that near (0,0),

$$f(x, y) = \frac{1}{2}(x, y) H \begin{pmatrix} x \\ y \end{pmatrix} + \text{higher order terms.}$$

Show that at the origin, H is also the matrix of $\hat{h}$ with respect to the basis $\{e_1 = \mathbf{i}, e_2 = \mathbf{j}\}$ of $T_{(0,0,0)} M$. (I.e., show that $\hat{h}(e_i, e_j) = (\partial^2 f / \partial x^i \partial x^j)(0,0)$.)

8. Let V be a finite-dimensional vector space, g an inner product on V, h an arbitrary bilinear form on V. Let $\mathbf{g}, \mathbf{h}$ be the linear maps $V \to V^*$ induced by g, h respectively; $\mathbf{g}$ is an isomorphism but $\mathbf{h}$ need not be. Define $S : V \to V$ by $S = \mathbf{g}^{-1} \circ \mathbf{h}$. Show that S is the unique endomorphism of V for which

$$h(X, Y) = g(X, S(Y))$$

for all $X, Y \in V$.

9. Let E be a vector bundle over M with connection ∇ and curvature F. Let ∇ also denote the induced connection on the endomorphism bundle $\mathrm{End}(E)$. Recall that the *Bianchi identity* asserts that $d_\nabla F = 0$, where F is viewed as an $\mathrm{End}(E)$-valued 2-form and where $d_\nabla : \Omega^2(\mathrm{End}(E)) \to \Omega^3(\mathrm{End}(E))$ is covariant exterior derivative. Show that the Bianchi identity is equivalent to the assertion that for all vector fields X, Y, Z on M, we have

$$\nabla_X(F(Y, Z)) + \nabla_Y(F(Z, X)) + \nabla_Z(F(X, Y)) = 0.$$

10. Notation as in problem 4a. Let C be the (image of) a geodesic in $(\mathbf{R}^2_+, g_1)$. Show that C is either (i) a subinterval of a vertical line or (ii) an arc lying in a semicircle whose endpoints are on the x-axis (i.e. a semicircle of the form $\{(x-a)^2 + y^2 = r^2, y \geq 0\}$).