Differential Geometry—MTG 6257—Spring 2000
Problem Set 2: Normal Coordinates

Notation for all the problems below. Unless otherwise specified, (M, g) is a Riemannian manifold of dimension n. Summation notation for repeated indices is used.

Definition 1. Let $p \in M$. A *normal neighborhood of* p is the diffeomorphic image, under $\exp_p$, of a ball $B_\epsilon(0) \subset T_pM$ for some $\epsilon > 0$ (the *radius* of U).

Definition 2. Let $p \in M$ and let U be a normal neighborhood of p. Let $\mathbf{e} = \{e_i\}_1^n$ be an orthonormal basis of T_pM. Let r be the radius of U. Define a diffeomorphism

$$\begin{aligned} \phi_{\mathbf{e}} : (B_r(0) \subset \mathbf{R}^n) &\rightarrow U \\ (x^1, \ldots, x^n) &\mapsto \exp_p(x^i e_i). \end{aligned}$$

Then $(U, \phi_{\mathbf{e}}^{-1})$ is a coordinate chart, and the functions $x^i \circ \phi_{\mathbf{e}}^{-1}$ are called (a system of) *normal coordinates* on U, centered at p.

Problems

1. Notation as in Definition 2.

(a) Show that

$$\frac{\partial}{\partial x^i}|_p = e_i.$$

(b) Show that at the center point p,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0.$$

(c) Let $|x| = (\sum (x^i)^2)^{1/2}$. Show that for $q \in U$, $|x|(q) = \mathrm{dist}(q, p)$.

2. Let U be a normal neighborhood of p, and let $\{x^i\}, \{y^i\}$ be two systems of normal coordinates on U centered at p. Show that there exists a constant orthogonal matrix A relating the two coordinate systems ($y^i = A^i{}_j x^j$).

3. In this problem you will use the Jacobi equation to show that the Riemann tensor determines the Taylor expansion of the metric in normal coordinates, and you will determine the first few terms of the expansion. Below, U is a normal neighborhood centered at p, B is the corresponding ball in T_pM, $\mathbf{e}$ is an orthonormal basis of T_pM, and $\{x^i\}$ are the corresponding normal coordinates on U.

Let $x = x^i e_i$ (so that $\exp_p(x)$ is exactly the point whose normal coordinates are $(x^1, \ldots, x_n)$), let $v = v^i e_i \in T_pM$, and let $\alpha(s,t) = \bar{\alpha}_s(t) = \exp_p(t(x + sv))$. For each s, $\bar{\alpha}_s$ is a geodesic, so α is a variation of α_0 through geodesics, and $V := \alpha_* \frac{\partial}{\partial s}$ is a Jacobi field along $\gamma := \bar{\alpha}_0$. Let $T = \gamma'$, and for any vector field W along γ, let $W' = \nabla_T W$.

(a) Check that $V(0) = 0, V'(0) = v$, and $V(1) = v^i \frac{\partial}{\partial x^i}|_{\exp_p(x)}$.

(b) Let $f(t) = \|V(t)\|^2$, so that

$$
\begin{aligned}
f'(t) &= 2(V, V') \\
f''(t) &= 2(V, V'') + 2(V', V'), \\
f'''(t) &= 2(V, V''') + 6(V', V''), \\
f''''(t) &= 2(V, V'''') + 8(V', V''') + 6(V'', V''),
\end{aligned}
$$

etc. Using the Jacobi equation $V'' = R(T, V)T$, show that the m^{th} derivative $f^{(m)}(t)$ can be computed from V, V', the Riemann tensor R, and the covariant derivatives $(\nabla_T)^i R$ up to order $m - 2$.

(c) With f as above, show that $f(0) = 0 = f'(0) = f'''(0)$ and that $f''(0) = 2\|v\|^2$, $f''''(0) = 8(R(T, v)T, v)|_0 = -8R_{ikjl}v^iv^jx^kx^l$, where $\{R_{ikjl}\}$ are the components of the Riemann tensor at p in the basis $\{e_i\}$. (There is no misprint in the order of the indices above; the $v's$ are paired with the first and third indices of R, and the x's with the second and fourth.) Hence show that

$$
\begin{aligned}
f(t) &= t^2\|v\|^2 - \frac{1}{3}t^4 R_{ikjl}v^iv^jx^kx^l + O(t^5\|v\|^2) \\
&= t^2 v^i v^j(\delta_{ij} - \frac{1}{3}t^2 R_{ikjl}v^iv^jx^kx^l + O(t^3)).
\end{aligned}
$$

(d) Using (a) and (c), show that the metric coefficients $g_{ij}(x) := (\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ satisfy

$$
g_{ij}(x) = \delta_{ij} - \frac{1}{3}R_{ikjl}x^kx^l + O(|x|^3).
$$

4. Let $\{x^i\}$ be local coordinates (not necessarily normal coordinates) on an open set $U \subset M$, let $\{e_i\}$ be an orthonormal basis of TM over U, and let $\{\theta^i\}$ be the basis of T^*M dual to $\{e_i\}$. Let A be the matrix-valued function relating the bases $\{dx^i\}, \{\theta^i\}$ of T^*M: $dx^i = A^i{}_j\theta^j$.

(a) Express $dx^1 \wedge \ldots \wedge dx^n$ in terms of A and $\theta^1 \wedge \ldots \wedge \theta^n$. (You have done this before.)

(b) Express the matrix $g_{..}$ of metric coefficients $g_{ij} = (\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ in terms of the matrix A.

(c) Assume that M is oriented and that $\{x^i\}$ is a positively oriented coordinate system. Show that the volume form vol_g can be expressed in these local coordinates by

$$
\text{vol}_g = \sqrt{\det(g_{..})}\, dx^1 \wedge \ldots \wedge dx^n.
$$

5. Assume M is oriented.

(a) Use the results from the preceding problems to show that in positively-oriented normal coordinates $\{x^i\}$,

$$\mathrm{vol}_g = (1 - \frac{1}{6}R_{kl}x^k x^l + O(|x|^3))dx^1 \wedge \ldots \wedge dx^n,$$

where $\{R_{kl}\}$ are the components of the Ricci tensor at p in the basis $\{dx^i\}$. (You will also want to use the formula derived in homework last semester for the directional derivatives of the determinant function $\det : M_n(\mathbf{R}) \to \mathbf{R}$.)

(b) Let $\{y^i\}$ be standard coordinates on $\mathbf{R}^n$, let S^{n-1} be the unit sphere with the induced metric, let $\omega \in \Omega^{n-1}(S^{n-1})$ be the volume form, and let $V_{n-1} = \int_{S^{n-1}} \omega$ (the volume of the sphere). Show that

$$\int_{S^{n-1}} y^i y^j \omega = \frac{1}{n}\delta_{ij}V_{n-1}.$$

(There is a way to do this that does not involve any trigonometric integrals.)

(c) Last semester you showed for homework that on the complement of the origin in $\mathbf{R}^n$, $dy^1 \wedge \ldots \wedge dy^n = r^{n-1}dr \wedge \tilde{\omega}$, where $\tilde{\omega} = \pi^*\omega$ is the pullback of ω via the radial projection $\pi : y \mapsto y/|y|$ ($r^{n-1}dr \wedge \tilde{\omega}$ is the n-dimensional version of the "$r\ dr\ d\theta$" formula for the area form on $\mathbf{R}^2$). Let $S_r^{n-1}(p) \subset M$ denote the sphere of radius r centered at p in M (i.e. the image under $\exp_p$ of the sphere $\{x^i e_i \mid |x| = r\} \subset T_pM$), where r is taken small enough that $S_r^{n-1}(p)$ lies in a normal neighborhood of p. Show that

$$\mathrm{Vol}(S_r^{n-1}(p)) = V_{n-1}(1 - \frac{1}{6n}\mathsf{R}(p)r^2 + O(r^3)),$$

where $\mathsf{R}(p)$ is the scalar curvature at p. This quantifies, in terms of volume, the statement "larger curvature means smaller spheres" and shows that the *scalar* curvature provides the dominant correction to the Euclidean formula.

(d) With r as above, let $B_r(p) \subset M$ denote the ball of radius r centered at p. Derive the analogous asymptotic expansion of $\mathrm{Vol}(B_r(p))$ (to order r^2 as above).

6. Let U be a normal neighborhood centered at $p \in M$, let $\{x^i\}$ be normal coordinates centered at p. Let $r = |x|$; note that $dr = (\sum x^i dx^i)/r$. Recall the Gauss Lemma proved in class: the "radial" geodesics $t \mapsto \exp_p(tv)$ ($v \in T_pM$) are orthogonal to the spheres $S_{r_0}^{n-1}(p)$ contained in a normal neighborhood centered at p (notation as in problem 5c).

(a) Use the facts above to prove that on $U - \{p\}$, the metric g can be written in "polar normal coordinates" as $dr^2 + g_r$, where for each fixed r, g_r is the pullback of some metric on S^{n-1}. (Example: $M = \mathbf{R}^2 = U$ with the standard metric, $p = 0$; the standard coordinates are normal coordinates centered at p. The metric can be written as $dr^2 + r^2 d\theta^2$. In this example, $g_r = r^2 d\theta^2$ [generally g_r will not be of the form (function of r)$\times$(fixed metric on sphere) unless (M, g) has a lot of symmetry]. Were radial lines not orthogonal to circles centered at the origin, there would be a term propotional to $dr \otimes d\theta + d\theta \otimes dr$ in the metric.)

(b) Let $g_{..}$ be the matrix of g in normal coordinates. Show that $g_{ij}(x)x^j = x^i$ (i.e. the vector with components $\{x^i\}$ is an eigenvector of $g_{..}$ with eigenvalue 1). Thus if we write $g_{ij}(x) = \delta_{ij} + h_{ij}(x)$, as in problem 3d, we have $h_{ij}(x)x^j = 0$. Check directly that the $O(|x|^2)$-term in the formula in problem 3d has this property.

7. Let $M = S^n$, embedded the standard way in $\mathbf{R}^{n+1}$. Let $\{y^i\}$ be standard coordinates on $\mathbf{R}^{n+1}$ and let p be the "north pole" $(0, \ldots, 0, 1)$; there is a natural identification of T_pM with the hyperplane $\{y^{n+1} = 1\} \subset \mathbf{R}^{n+1}$. Let $\{e_i\}$ be the orthonormal basis of T_pM agreeing with $\partial/\partial y^i$ for $1 \leq i \leq n$. Let $r : M \to \mathbf{R}$ denote distance to p.

(a) Show that r is the "latitude-from-the-north-pole" angle: $r(q) = \cos^{-1}(p \cdot q)$, where on the right-hand side of this formula the points $p, q \in S^n$ are viewed as unit vectors in $\mathbf{R}^{n+1}$.

(b) Let $U = S^n - \{\text{south pole}\}$ and let $\{x^i\}$ be the normal coordinates on U determined by $\{e_i\}$. Express the $\mathbf{R}^{n+1}$-coordinate functions $y^i : S^n \to \mathbf{R}$ $(1 \leq i \leq n+1)$ in terms of the x^i.

(c) Show that in "polar normal coordinates" the metric on S^n takes the form $dr^2 + f(r)^2 g_0$ (see problem 6a), where g_0 is the standard metric on S^{n-1}. Give $f(r)$ explicitly. (You can do this problem by explicit computation, using part (b), unless you see a clever way to avoid the computation.)

(d) Recall that the Riemann tensor on S^n satisfies $R_{ijkl} = g_{ik}g_{jl} - g_{il}g_{jk}$. Use this to compute the scalar curvature of S^n.

(e) Using part (c), show that the volumes of $S_r^{n-1}(p) \subset S^n$ and $B_r(p) \subset S^n$ are of the form $V_{n-1}h(r)$ (notation as in problems 5bc), and in each case give the Taylor expansion of $h(r)$ to order r^2. Using part (d), check that your expansions agree with your answers to problems 5cd.