Differential Geometry—MTG 6257—Spring 2000
Problem Set 3: Lie Groups and related topics

Notation for all the problems below. Unless otherwise specified, G and H are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Identity elements of groups are denoted e. Summation notation for repeated indices is used.

1. Assume G is connected, and let $\tilde{G}$ be the universal covering space of G, with $\pi : \tilde{G} \to G$ the covering map. Recall that in class we defined a multiplication map $\tilde{G} \times \tilde{G} \to \tilde{G}$ as follows. Fix $\tilde{e} \in \pi^{-1}(e)$. Given $\tilde{g}_i \in \tilde{G}$ $(i = 1, 2)$, let $\tilde{\gamma}_i : [0, 1] \to \tilde{G}$ be a path in $\tilde{G}$ from $\tilde{e}$ to $\tilde{g}_i$. Let $g_i = \pi(\tilde{g}_i)$, $\tilde{\gamma}_i(t) = \pi(\gamma_i(t))$, and let $\gamma(t) = \gamma_1(t)\gamma_2(t)$. Let $\tilde{\gamma}$ be the unique lift of γ to $\tilde{G}$ with $\tilde{\gamma}(0) = \tilde{e}$. Then we declare $\tilde{g}_1\tilde{g}_2 = \tilde{\gamma}(1)$. In class we proved that this construction is independent of the choices of $\tilde{\gamma}_1, \tilde{\gamma}_2$, so that the multiplication map is well-defined.

(a) Prove that $\tilde{G}$ is a group with respect to this multiplication map, with identity $\tilde{e}$.

(b) Prove that the kernel of π is contained in the center of $\tilde{G}$. (The *center* of group is the subgroup consisting of all elements that commute with every group element.)

2. (a) Explicitly identify the subspaces of $M_n(\mathbf{R})$ corresponding to $\mathfrak{sl}(n, \mathbf{R})$, the Lie algebra of $SL(n, \mathbf{R})$.

Let $M_n(\mathbf{C})$ be the space of $n \times n$ complex matrices. Explicitly identify the subspaces of $M_n(\mathbf{C})$ corresponding to the Lie algebras of (b) $U(n)$, (c) $SU(n)$, and (d) $Sp(n)$. (These Lie algebras are denoted $\mathfrak{u}(n), \mathfrak{su}(n)$, and $\mathfrak{sp}(n)$ respectively.)

3. Let $\rho : G \to H$ be a Lie-group homomorphism, let $\dot{\rho} : \mathfrak{g} \to \mathfrak{h}$ be the induced Lie-algebra homomorphism, and let $K \subset G$, $\mathfrak{k} \subset \mathfrak{g}$ be the kernels of $\rho, \dot{\rho}$ respectively.

(a) Prove that K is a Lie group. (Don't spend time proving that K is a group; it's the "Lie" part that I want you to prove. I know that you know how to prove that the kernel of a group-homomorphism is a group.)

(b) Prove that $\mathfrak{k}$ is the Lie algebra of K.

(c) Prove that $\mathfrak{k}$ is an ideal in $\mathfrak{g}$.

4. Let M be a manifold, E a distribution on M. Let $E^\perp \subset T^*M$ be the subbundle defined by $E_p^\perp = \{\theta \in T_p^*M \mid \theta(X) = 0 \ \forall X \in E_p\}$ (the annihilator of E_p). Let $\mathcal{I} \subset \Omega^*(M)$ be the ideal generated by $\Gamma(E^\perp)$ (i.e. $\mathcal{I}$ is the space of linear combinations of differential forms of the form $\omega \wedge \theta$, where $\theta \in \Gamma(E^\perp)$.) Let $d\mathcal{I}$ be the image of $d : \mathcal{I} \to \Omega^*(M)$. Prove that E is involutive if and only if $d\mathcal{I} \subset \mathcal{I}$.

5. Calculate the exponential map explicitly for (a) $SU(2)$, (b) $SO(3)$, and (c) $SL(2, \mathbf{R})$. (All of these are three-dimensional Lie groups. In each case, choose a way of writing a typical Lie-algebra element X in terms of three real parameters, and then give an explicit formula for $\exp(X)$ that does not involve an infinite series. From class, we know that $\mathfrak{so}(3)$ consists of the skew-symmetric 3×3 matrices; use the relevant parts of problem 2 above for the other two Lie algebras).

6. Recall the "polar decomposition" of an invertible matrix from undergraduate linear algebra: any invertible matrix A can be written as a product RS, where R is an orthogonal matrix and S is a symmetric positive-definite matrix. Here S is the unique positive-definite square root of $A^t A$; this depends continuously on A, and hence so does R.

(a) Use the polar decomposition to show that there is a strong deformation retraction from $SL(2, R)$ to $SO(2)$. (A *strong deformation retraction* from a topological space A to a subspace B is a continuous map $A \to B$ that is homotopic to id_A through maps that restrict to the identity on B.) Use this to show that $SL(2, \mathbf{R})$ is connected.

(b) Show that the exponential map $\mathfrak{sl}(2, \mathbf{R}) \to SL(2, \mathbf{R})$ is not surjective. Thus $SL(2, \mathbf{R})$ is an example of a connected Lie group for which the exponential map is not surjective (its image still *generates*, but is not *equal* to the group).

7. Let G be any group acting transitively on a set S from the left.

(a) Show that for any two elements $p_1, p_2 \in S$, the stabilizers of p_1 and p_2 are conjugate (i.e. letting H_i denote the stabilizers, there exists $g \in G$ such that $H_2 = gH_1g^{-1}$, the image of H_1 under Ad_g).

(b) Let $p_0 \in S$. Show that the choice of p_0 determines a natural bijection $S \leftrightarrow G/H$, where $H = \mathrm{Stab}(p_0)$.

8. (a) Let E be a tensor bundle over G, and let $S \in E_e$ be Ad-invariant. Prove that the left-invariant and right-invariant extensions of S are bi-invariant and equal to each other. Deduce as a corollary that G admits a bi-invariant Riemannian metric iff $\mathfrak{g}$ admits an Ad-invariant inner product.

(b) Suppose h_e is an Ad-invariant inner product on $T_eG = \mathfrak{g}$. Show that for all $X, Y, Z \in \mathfrak{g}$, we have $h_e(\mathrm{ad}_X(Y), Z) = -h_e(Y, \mathrm{ad}_X(Z))$.

(c) With h_e as in (b), define $\omega_e \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}^*$ by $\omega_e(X, Y, Z) = h_e(X, [Y, Z])$. Show that ω_e lies in the totally alternating subspace $\bigwedge^3 \mathfrak{g}^*$. Hence show that ω extends to a bi-invariant 3-form on G.

9. The *center* of $\mathfrak{g}$ is $\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \ \forall Y \in \mathfrak{g}\}$. The center of G is $Z(G) = \{g \in G \mid gh = hg \ \forall h \in G\}$. Prove that $Z(G)$ is a closed Lie subgroup of G with Lie algebra $\mathfrak{z}(\mathfrak{g})$. Deduce that $Z(G)$ is 0-dimensional (equivalently, discrete) iff $\mathfrak{z}(\mathfrak{g})$ is trivial (i.e. $\{0\}$).

10. (On next page)

10. In class we proved that if G is compact, then G admits a bi-invariant Riemannian metric. In this problem you will prove a partial converse to this fact. Each part below uses the preceding part. If there's any part you can't do, assume it and move on to the next part.

Assume that G admits a bi-invariant metric h, and let ∇ be the Levi-Civita connection and R the Riemann curvature tensor determined by h.

(a) Prove that if X and Y are left-invariant vector fields, then $\nabla_X Y = \frac{1}{2}[X, Y]$. Deduce as a corollary that the integral curves of a left-invariant vector field are geodesics. (Thus the exponential map in Riemannian geometry is a generalization of the exponential map on a Lie group admitting a bi-invariant metric.)

(b) Show that if X, Y, and Z are left-invariant vector fields, then $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$.

(c) Let X and Y be left-invariant vector fields. Show that at any point $g \in G$, we have $h(R(X, Y)Y, X) = \frac{1}{4}\|[X, Y]|_e\|^2$. Deduce from this that the sectional curvatures of G are all non-negative.

(d) Let Ric be the Ricci tensor. Show that if $X \in \mathfrak{g} = T_eG$ and X is not in the center of $\mathfrak{g}$, then $Ric(X, X) > 0$.

(e) Show that if the center of $\mathfrak{g}$ is trivial (i.e. $\mathfrak{z}(\mathfrak{g}) = \{0\}$), then there exists $c > 0$ such that for all unit vectors $v \in TG$, we have $Ric(v, v) \geq c$.

(f) Use Myers' Theorem to conclude that if G admits a bi-invariant metric and the center of G is 0-dimensional (equivalently, if $\mathfrak{g}$ admits an Ad-invariant inner product and the center of $\mathfrak{g}$ is trivial), then G is compact.