Differential Geometry II—MTG 6257—Spring 2013
Problem Set 1

1. Let (M, g) be the sphere (S^n, g_{std}), where g_{std} (henceforth simply g) is the Riemannian metric inherited by restriction from the standard Riemannian metric g_{Euc} on $\mathbf{R}^{n+1}$.

(a) Let $x \in S^n$. Under the canonical identification of $T_x\mathbf{R}^{n+1}$ with $\mathbf{R}^{n+1}$, with what subspace of $\mathbf{R}^{n+1}$ is T_xS^n identified (in terms of x)?

(b) Recall that the canonical identification of each tangent space of $\mathbf{R}^{n+1}$ with $\mathbf{R}^{n+1}$ gives us an identification of {vector fields on $\mathbf{R}^{n+1}$} with {$\mathbf{R}^{n+1}$-valued functions on $\mathbf{R}^{n+1}$}. This identification gives meaning to the notion of "constant vector field" on $\mathbf{R}^{n+1}$, namely a vector field that corresponds to a constant $\mathbf{R}^{n+1}$-valued function. For each $v \in \mathbf{R}^{n+1}$, let $\tilde{Y}^{(v)}$ be the corresponding "constant" vector field on $\mathbf{R}^{n+1}$, and let $Y^{(v)}$ be the (tangent) vector field on S^n defined by $Y^{(v)}(x) := Y^{(v)}_x := \pi_x(\tilde{Y}^{(v)}_x)$, where $\pi_x : T_x\mathbf{R}^{n+1} \to T_xS^n$ is orthogonal projection.

For each $v \in \mathbf{R}^{n+1}$, also define a function $f_v : S^n \to \mathbf{R}$ by $f_v(x) = v \cdot x$ (ordinary dot-product).

Recall that for any real-valued function f on a Riemannian manifold (N, h), the *gradient* of f, denoted grad f, is the vector field on N defined at each $p \in N$ to be the unique vector satisfying

$$h_p((\text{grad } f)|_p, u) = \langle df|_p, u\rangle \quad \forall u \in T_pN.$$

For the Riemannian manifold (S^n, g), show that, for each $v \in \mathbf{R}^{n+1}$,

$$\text{grad } f_v = Y^{(v)}$$

(c) Let ∇ be the Levi-Civita connection on (S^n, g). Let $v, w \in \mathbf{R}^{n+1}$, and let $V = Y^{(v)}$, $W = Y^{(w)}$. Compute $\nabla_V W$ explicitly as an algebraic expression in terms of V, W, f_v, and f_w. (Not all of these four objects may enter your formula. "Algebraic expression" here means something whose value at each $x \in S^n$ is given explicitly by the values $V_x, W_x, f_v(x)$, and/or $f_w(x)$.)

(d) Using part (c) and the fact that ∇ is torsion-free, express the Lie bracket $[V, W]$ explicitly as an algebraic expression in terms of V, W, f_v, and f_w (where V, W are as in (c)).

(e) Let $z \in \mathbf{R}^{n+1}$, let $Z = Y^{(z)}$, let V, W be as above, and let K be the curvature of ∇. Using the calculations above, show that

$$K(V, W)(Z) = g(W, Z)V - g(V, Z)W.$$

2. Let ∇ be a connection on a vector bundle E over a manifold M. The connection ∇ is called *flat* if its curvature K is identically zero (i.e. if $K(X,Y)(s) = 0 \in \Gamma(E)$ for all vector fields X, Y on M and all sections s of E).

Let M be an arbitrary manifold and let E be the product bundle $M \times \mathbf{R}^k$ over M. In class we defined a connection ∇ on E by identifying sections s of E with $\mathbf{R}^k$-valued functions f_s, and setting $(\nabla_X s)_p = X_p(f_s)$. Show that this connection is flat.

3. Let E be a vector bundle over a manifold M, and let $s \in \Gamma(E)$. Show that s is an embedding of the manifold M into the manifold E.

4. Let E be a vector bundle of rank k over a manifold M, and let ∇ be a connection on E. Let $\mathrm{Mat}_{k\times k}(\mathbf{R})$ denote the space of all $k \times k$ matrices with real entries, and recall that $GL(k, \mathbf{R})$ is the set of all invertible such matrices, an open subset of $\mathrm{Mat}_{k\times k}(\mathbf{R})$

Let $U \subset M$ be open, and assume that $E|_U$ has a basis of sections $\{s_1, \ldots, s_k\}$. Let $\{s'_1, \ldots, s'_k\}$ be another basis of sections of $E|_U$. Necessarily, the second basis is related to the first basis by

$$s'_\mu = \sum_{\nu=1}^{k} s_\nu G^\nu{}_\mu, \quad 1 \le \mu \le k,$$

for a unique, smooth function $G : U \to GL(k, \mathbf{R}) \subset \mathrm{Mat}_{k\times k}(\mathbf{R})$. (At each $p \in U$, the $G^\nu{}_\mu(p)$ are the entries of $G(p)$.)

Let Θ, Θ' be the connection forms of ∇ relative to the bases $\{s_1, \ldots, s_k\}$ and $\{s'_1, \ldots, s'_k\}$, respectively. Show that

$$\Theta' = G^{-1}\Theta G + G^{-1}dG, \tag{1}$$

where G^{-1} and G are treated as matrices whose entries are real-valued functions; Θ, Θ', and dG are treated as matrices whose entries are real-valued 1-forms; and $(dG)^\mu{}_\nu = d(G^\mu{}_\nu)$. Helpful observation: (1) is equivalent to

$$\Theta'(X) = G^{-1}\Theta(X)G + G^{-1}X(G) \quad \forall X \in \Gamma(TM|_U). \tag{2}$$

In (2), all of the objects $\Theta'(X), \Theta(X), G^{-1}, G$, and $X(G)$ may be viewed either as $\mathrm{Mat}_{k\times k}$-valued functions, or as matrices whose entries are real-valued functions. In the former point of view, at each $p \in U$, $X_p(G)$ is the directional derivative of the $\mathrm{Mat}_{k\times k}$-valued function G in the direction $X_p \in T_pM$; in the latter point of view, $X_p(G)$ is a matrix whose $(\mu,\nu)^{\text{th}}$ entry is $X_p(G^\mu{}_\nu)$.