Differential Geometry II—MTG 6257—Spring 2013
Problem Set 2

1. Let M be the open ball of radius 1 in $\mathbf{R}^n$, centered at the origin. Let $r : \mathbf{R}^n \to \mathbf{R}$ denote distance to the origin. Let g_{Euc} be the standard Riemannian metric on $\mathbf{R}^n$, restricted to the open set M. Then define a metric g on M by

$$g = \frac{4}{(1-r^2)^2} g_{\mathrm{Euc}} \ .$$

Show that M has constant curvature -1. This Riemannian manifold is called the *Poincaré disk* or (the Poincaré model of) *hyperbolic n-space.*

2. Let (M, g) be a Riemannian manifold. Let c be a positive constant and let $g_{\mathrm{rescaled}} = c^2 g$.

(a) Show that the Riemann tensors of (M, g) and $(M, g_{\mathrm{rescaled}})$ are identical. (Here, "Riemann tensor" means the "$R(X, Y)Z$" version, not the "$g(R(X, Y)Z, W)$" version.)

(b) Let σ and $\sigma_{\mathrm{rescaled}}$ be the sectional-curvature functions of (M, g) and $(M, g_{\mathrm{rescaled}})$. Show that $\sigma_{\mathrm{rescaled}} = c^{-2}\sigma$.

Note: You may be wondering why I bothered to write the rescaling factor as c^2 rather than just c. The reason is that if g is rescaled by c^2, then distances are rescaled by c, which is more in keeping with the usual meaning of "rescaling". The sphere of radius c in $\mathbf{R}^{n+1}$ is isometric to the unit sphere with metric $c^2 g_{\mathrm{std}}$, where g_{std} is the standard metric on S^n.