Differential Geometry II—MTG 6257—Spring 2013
Problem Set 3

1. Let (M,g) be Riemannian manifold, let $\{e_i\}_1^n$ be a local basis of sections of TM, defined on $U \subset M$, and let Θ be the connection form of the Levi-Civita connection with respect to this local basis. Let $\{\theta^i\}$ be the dual basis of sections of T^*M on U.

(a) Show that

$$d\theta^i = -\Theta^i{}_j \wedge \theta^j, \quad 1 \le i \le n. \tag{1}$$

(In case you have forgotten the following useful identity: for any real-valued 1-form ω and vector fields X, Y, we have $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$.)

(b) Assume now that $\{e_i\}_1^n$ is orthonormal (at each point of U). Recall that this implies that Θ, viewed as a matrix of real-valued 1-forms, is antisymmetric. Show that Θ is the *unique* antisymmetric matrix of real-valued 1-forms such that (1) holds.

2. Let (M,g) be a Riemannian manifold, $U \subset M$, and suppose that $\{\theta^i\}_{i=1}^n$ are 1-forms on U such that $g = \sum_{i=1}^n \theta^i \otimes \theta^i$. (For example, the Euclidean metric on $\mathbf{R}^n$ can be written as $\sum_i dx^i \otimes dx^i$.) Show that (a) the set $\{\theta^i|_p\}$ is linearly independent, and hence a basis of T_p^*U, for all $p \in U$, and that (b) that if $\{e_i\}_1^n$ is the dual basis of local sections of TM, then $\{e_i\}$ is orthonormal (at each point).

Hence whenever g is expressed in the form $g = \sum_{i=1}^n \theta^i \otimes \theta^i$, the set $\{\theta^i\}$ is automatically dual to an orthonormal basis of local sections of TM. This, combined with problem 1, often gives a faster way of finding a connection form Θ than by computing covariant derivatives or Christoffel symbols. (This is a hint to help you do problem 3c efficiently.)

3. **Surfaces.** Let (M,g) be a two-dimensional Riemannian manifold. Let $\{e_1, e_2\}$ be a local orthonormal basis of TM, defined on $U \subset M$, and let Θ be the connection form of the Levi-Civita connection with respect to this local basis.

(a) Show that the matrix-valued 2-form representing the Riemann tensor in this basis is

$$\begin{pmatrix} 0 & d\Theta^1{}_2 \\ -d\Theta^1{}_2 & 0 \end{pmatrix}. \tag{2}$$

(b) Let $\{\theta^1, \theta^2\}$ be the local basis of T^*M dual to $\{e_1, e_2\}$. Then $d\Theta^1{}_2 = K\theta^1 \wedge \theta^2$ for some unique function $K : U \to \mathbf{R}$. Show that, in terms of the components of the Riemann tensor in the given bases, $K = R^1{}_{212} = R_{1212}$. From this, show that at each $p \in U$, $K(p)$ is the sectional curvature of M in the (unique) 2-dimensional subspace of T_pM. In particular, K is independent of the choice of local bases, and the construction above yields a well-defined function $K : M \to \mathbf{R}$. This function $K : M \to \mathbf{R}$ is called the *Gaussian curvature*. (Note: the scalar-valued function K in this problem is not the same as what we used this letter for in class, which was the endomorphism-valued curvature two-form of a general connection. "K" just happens to be the letter most commonly used for Gaussian curvature.)

(c) Let $\mathcal{H}^2 = \{(x, y) \in \mathbf{R}^2 \mid y > 0\}$. Define a metric g_{hyp} on $\mathcal{H}$ by

$$g_{\mathrm{hyp}} = \frac{1}{y^2} g_{\mathrm{Euc}} \quad = \quad \frac{dx \otimes dx + dy \otimes dy}{y^2} = \left(\frac{dx}{y}\right) \otimes \left(\frac{dx}{y}\right) + \left(\frac{dy}{y}\right) \otimes \left(\frac{dy}{y}\right).$$

Show that $(\mathcal{H}^2, g_{\mathrm{hyp}})$ has constant curvature -1. This Riemannian manifold is called the *upper-half-plane model* of hyperbolic 2-space.

Remark. For $n > 2$, we can analogously define $\mathcal{H}^n = \{(x^1, \ldots, x^n) \in \mathbf{R}^n \mid x^n > 0\}$, and the metric

$$g_{\mathrm{hyp}} = \frac{1}{(x^n)^2} g_{\mathrm{Euc}}$$

on $\mathcal{H}^n$. The manifold $(\mathcal{H}^n, g_{\mathrm{hyp}})$ has constant curvature -1, and is called the *upper-half-space model* of hyperbolic n-space. The method used to compute the curvature for the case $n = 2$ can be used for general $n \geq 2$, but only for $n = 2$ does the "$\Theta \wedge \Theta$" term in the "$d\Theta + \Theta \wedge \Theta$" expression for the curvature vanish as it did in problem 3a.

4. Let (M, g) be a Riemannian manifold, $I \subset \mathbf{R}$ an interval, and $\gamma : I \to M$ a smooth curve for which $\gamma'(t)$ is nowhere zero and such that

$$\nabla_{\gamma'} \gamma' = f \gamma'$$

for some function $f : I \to \mathbf{R}$. Show that γ can be reparametrized as a geodesic. I.e. show that there exists an interval J and a diffeomorphism $\phi : J \to I$ such that $\gamma \circ \phi$ is a geodesic. (Hint: just as in Calculus 3, any curve with nonvanishing velocity can be reparametrized by arclength.)

5. Let $(\mathcal{H}^2, g)$ be as in problem 3c.

(a) Let $x_0 \in \mathbf{R}$, and let C be an open semicircle in the upper half-plane centered at $(x_0, 0)$ (i.e. $\{(x, y) \in \mathcal{H}^2 \mid (x - x_0)^2 + y^2 = R^2\}$ for some $R > 0$). Choose a parametrization γ of C. Show that γ can be reparametrized as a geodesic.

(b) Same as part (a), but for the vertical ray $C = \{(x_0, y) \mid y > 0\}$.

Remark. It is easy to see that given a point p in the upper half-plane, and a non-vertical straight line ℓ through (x_1, y_1), there exists a unique circle centered on the x-axis that is tangent to ℓ at p. It follows that the image of every geodesic in $(\mathcal{H}^2, g_{\mathrm{hyp}})$ has image lying in one of the semicircles or vertical rays considered above.