Differential Geometry II—MTG 6257—Spring 2013
Problem Set 4

1. Recall the "polar decomposition" of an invertible matrix from undergraduate linear algebra: any invertible matrix A can be written as a product RS, where R is an orthogonal matrix and S is a symmetric positive-definite matrix. If we choose S to be the unique positive-definite square root of A^tA, which we henceforth do, then S depends continuously on A (you are allowed to assume this), and hence so does R.

Use the polar decomposition to show *any one* of the following (you do not have to show more than one):

1. There is a strong deformation retraction from $SL(2,\mathbf{R})$ to $SO(2)$. (A *strong deformation retraction* from a topological space A to a subspace B is a continuous map $A \to B$ that is homotopic to id_A through maps that restrict to the identity on B. Choose this option only if you are already familiar with the terminology.)

2. $SL(2,\mathbf{R})$ is homeomorphic to $S^1 \times \mathbf{R}^2$. (Note: do not confuse **homeo**morphism with **homo**morphism. The product group $S^1 \times \mathbf{R}^2$ is not isomorphic to $SL(2,\mathbf{R})$.)

3. $SL(2,\mathbf{R})$ is diffeomorphic to $S^1 \times \mathbf{R}^2$.

Note: the third assertion obviously implies the second assertion, which implies the first. The first is the easiest to show, *if* you are familiar with the terminology. For some approaches to the second assertion, "diffeomorphic" will be no harder to show than "homeomorphic". But if your approach just shows that the two manifolds are homeomorphic, that is sufficient for this problem.

2. Recall that the Lie algebra of $SL(2,\mathbf{R})$ is $\mathfrak{sl}(2,\mathbf{R}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a,b,c \in \mathbf{R} \right\}$.

(a) Show that if $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, then $X^2 = \Delta I$, where $\Delta = a^2 + bc = -\det(X)$.

(b) Use part (a) to compute explicit formulas for $\exp(X)$ in the cases $\Delta > 0$, $\Delta = 0$, $\Delta < 0$, where the notation is as in part (a). To help you check that you're on the right track: for $\Delta > 0$, you should find that

$$\exp(X) = (\cosh\sqrt{\Delta})I + \frac{\sinh\sqrt{\Delta}}{\sqrt{\Delta}}X. \tag{1}$$

(c) From your explicit formulas in (b), show that for all $X \in \mathfrak{sl}(2,\mathbf{R})$, $\mathrm{tr}(\exp(X)) \geq -2$.

(d) Find $A \in SL(2, \mathbf{R})$ with $\mathrm{tr}(A) < -2$, and hence show that $SL(2, \mathbf{R})$ is a connected Lie group for which the exponential map is not surjective. (Connectedness follows from problem 1.)

3. Another Lie group for which it is easy to compute the exponential map explicitly is $SO(3)$. Using the same general procedure as in problem 2ab, compute $\exp : \mathfrak{so}(3) \to SO(3)$ (you will not need as many cases as for $SL(2, \mathbf{R})$). From your explicit formula, deduce that all one-parameter subgroups ($t \mapsto \exp(tX)$) in $SO(3)$ are periodic.

Remark: The fact that exp is easy to compute for $SL(2, \mathbf{R})$ and $SO(3)$ is a feature of low-dimensionality. But the reason that the computations for these two particular groups are so similar to each other, and not just easy, is deeper: the Lie algebras $\mathfrak{sl}(2, \mathbf{R})$ and $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ are *real forms* of the same complex Lie algebra, $\mathfrak{sl}(2, \mathbf{C})$. (A real Lie algebra $\mathfrak{g}$ can be *complexified* to a complex Lie algebra $\mathfrak{g}_{\mathbf{C}} = \{X + iY \mid X, Y \in \mathfrak{g}\}$, where the bracket on $\mathfrak{g}_{\mathbf{C}}$ is defined the "obvious" way, using the distributive law and treating i^2 as -1. We call $\mathfrak{g}$ a *real form* of a complex Lie algebra $\mathfrak{g}'$ if $\mathfrak{g}_{\mathbf{C}}$ is isomorphic, as a complex Lie algebra, to $\mathfrak{g}'$.)

4. Let $G = \{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, z > 0\} \subset GL(2, \mathbf{R})$ (the sign-restriction on x and z is just to make G connected, so that this problem can be seen as a companion to problem 5 below). Writing the general element of G as above, we may view x, y, and z as functions $G \to \mathbf{R}$. With this understanding, define three-forms μ_L, μ_R on G by

$$\mu_L = \frac{dx \wedge dy \wedge dz}{x^2 z},$$

$$\mu_R = \frac{dx \wedge dy \wedge dz}{x z^2}.$$

Show that μ_L is left-invariant and that μ_R is right-invariant. Use this to deduce that G does not admit a bi-invariant volume form.

Some hints: (i) When showing that, say, $L_g^* \mu_L = \mu_L$, do not use the letters x, y, z for the entries of g; otherwise the "x" in g will mean something different from the "x" in dx (etc. for y, z). Instead, take g to be a fixed but arbitrary element $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ of G. (ii) From general facts about pullbacks, $L_g^* \frac{dx \wedge dy \wedge dz}{x^2 z} = \frac{L_g^* dx \wedge L_g^* dy \wedge L_g^* dz}{(L_g^* x)^2 L_g^* z} = \frac{d(L_g^* x) \wedge d(L_g^* y) \wedge d(L_g^* z)}{(L_g^* x)^2 L_g^* z}$.

5. Define functions $x, y, z, w : SL(2, \mathbf{R}) \to \mathbf{R}$ by writing the general element of $SL(2, \mathbf{R})$ as $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. These four functions are not independent of each other; they satisfy the equation $xw - yz = 1$ identically.

Define $U_1 = \{g \in SL(2, \mathbf{R}) \mid x(g) \neq 0\}$, and analogously define U_2, U_3, U_4 (with $y \neq 0$ on U_2, $z \neq 0$ on U_3, and $w \neq 0$ on U_4). These four open sets cover $SL(2, \mathbf{R})$.

Define a 3-form on each of these sets as follows:

$$\begin{aligned}
\mu_1 &= \frac{dx \wedge dy \wedge dz}{x} \quad \text{on } U_1 \ , \\
\mu_2 &= \frac{dx \wedge dy \wedge dw}{y} \quad \text{on } U_2 \ , \\
\mu_3 &= -\frac{dx \wedge dz \wedge dw}{z} \quad \text{on } U_3 \ , \\
\mu_4 &= -\frac{dy \wedge dz \wedge dw}{w} \quad \text{on } U_4 \ .
\end{aligned}$$

(a) Show that $\mu_i = \mu_j$ on $U_i \bigcap U_j$, $1 \le i < j \le 4$. *Note*: You can do this with three computations (the cases with $i = 1$) rather than six, by using the fact that each μ_i is continuous on its domain, and that $U_1 \bigcap U_i \bigcap U_j$ is dense in $U_i \bigcap U_j$ for $2 \le i < j \le 4$.

(b) In view of part (a), we can define a volume form μ on $SL(2, \mathbf{R})$ by setting $\mu = \mu_i$ on U_i. Show that μ is bi-invariant.

Suggestions: (i) Use the hints from problem 4, with appropriate changes. (ii) Show that $L_g^* \mu_1 = \mu_1$ on $U_1 \bigcap L_g^{-1}(U_1)$. Then, instead of doing 15 more computations of the same type to show $L_g^* \mu_i = \mu_j$ on $U_j \bigcap L_g^{-1}(U_i)$ for all i, j, or just asserting that "the other cases are similar" without actually checking, use a continuity argument based on denseness of the U_i (applied both to the domains of the μ_i and to g) to show that this one computation implies that $L_g^* \mu = \mu$ globally, for all $g \in SL(2, \mathbf{R})$.