$\mathcal{F}$-linearity, tensoriality, and related notions

- Throughout, M is an arbitrary manifold and $n := \dim(M)$.
- For any vector bundle E over M:
  1. $\pi_E : E \to M$ denotes the projection map.
  2. For each $p \in M$, $E_p := \pi_E^{-1}(p)$, the fiber of E over p.
  3. We use the terminology *set-theoretic section of* E for any map $s : M \to E$, not necessarily smooth (or even continuous), such that $\pi_E \circ s = \mathrm{id}_M$. We reserve the terminology *section of* E for a *smooth* set-theoretic section.
  4. $\Gamma(E)$ denotes the space of sections of E.
  5. For $s \in \Gamma(E)$, the value of s at p may be denoted $s(p)$, s_p, or $s|_p$.
  6. For $U \subset M$ open and $s \in \Gamma(E|_U)$, the *extension of s by 0 to M* is the set-theoretic section $\tilde{s} : M \to E$ such that
  $$\tilde{s}(p) = \begin{cases} s(p) & \text{if } p \in U, \\ 0 & \text{if } p \notin U. \end{cases}$$
  7. Let $p \in M$, $v \in E_p$, and $s \in \Gamma(E)$. We say that s *is an extension of* v, or that s *extends* v, if $s(p) = v$.
  8. If $\kappa = \mathrm{rank}(E) > 0$, then for $U \subset M$ open, a *basis of sections of E over U*, or *basis of sections of $E|_U$*, will mean an ordered κ-tuple $\{s_\mu \in \Gamma(E|_U)\}_{\mu=1}^{\kappa}$ such that for all $p \in U$, $\{s_\mu(p)\}$ is a basis of E_p. (This is an abuse of terminology, but is convenient.)
- We write $\mathcal{F} = \mathcal{F}(M) = C^\infty(M)$ (the algebra of smooth functions $M \to \mathbf{R}$). For any vector bundle E over M, there is a natural action of $\mathcal{F}$ on $\Gamma(E)$: for $s \in \Gamma(E)$ and $f \in \mathcal{F}$, we define $fs \in \Gamma(E)$ by $(fs)(p) = f(p)s(p)$ (it is easily seen that the set-theoretic section fs is smooth). Thus the vector space $\Gamma(E)$ is canonically an $\mathcal{F}$-module.

1 $\mathcal{F}$-linearity and tensoriality

In this section of these notes, E and F denote fixed, arbitrary vector bundles over M.

Definition 1.1 Let $L : \Gamma(E) \to \Gamma(F)$ be a map.

1. We say that L is *$\mathcal{F}$-linear* if $L(s_1 + s_2) = L(s_1) + L(s_2)$ and $L(fs) = fL(s)$ for all $s_1, s_2, s \in \Gamma(E)$ and all $f \in \mathcal{F}$. (Thus every $\mathcal{F}$-linear map is linear.) Equivalent definitions are:
   - L is *$\mathcal{F}$-linear* if L is linear and $L(fs) = fL(s)$ for all $s \in \Gamma(E)$ and all $f \in \mathcal{F}$.
   - An *$\mathcal{F}$-linear map* $\Gamma(E) \to \Gamma(F)$ is a homomorphism of $\mathcal{F}$-modules.

2. We say that L is *tensorial* if there exists a bundle homomorphism $H : E \to F$, covering the identity map id_M, such that for all $s \in \Gamma(E)$,
$$L(s) = H \circ s. \tag{1}$$

3. For $s \in \Gamma(E)$ and $p \in M$, we say that $L(s)|_p$ *depends only of value of s at p* if for all $s_1 \in \Gamma(E)$ with $s_1(p) = s(p)$, we have $L(s)|_p = L(s_1)|_p$. In these notes, we will say that L *is determined by 0-jets* if for all $s \in \Gamma(E)$ and $p \in M$, $L(s)|_p$ depends only of value of s at p.[1]

Observe that there is a natural one-to-one correspondence

$$\{\text{bundle homomorphisms } E \to F\} \quad \longleftrightarrow \quad \Gamma(\mathrm{Hom}(E, F)), \tag{2}$$
$$H \quad \longleftrightarrow \quad \hat{H}. \tag{3}$$

Specifically, given a homomorphism $H : E \to F$ and $p \in M$, the map $H|_{E_p}$ is a linear map $\hat{H}_p : E_p \to F_p$, i.e. an element of the fiber $\mathrm{Hom}(E, F)_p$. Smoothness of H implies smoothness of $\hat{H}$ (proof left to reader). Thus the map $p \mapsto \hat{H}_p$ is a section of $\mathrm{Hom}(E, F)$. Conversely, given $\hat{H} \in \Gamma(\mathrm{Hom}(E, F))$, we can define a map $H : E \to F$ by $H(v) = \hat{H}_{\pi_E(v)}(v)$ for all $v \in E$. By definition, $\hat{H}_p$ is a (linear) map $E_p \to F_p$ for all p, so $\pi_F(H(v)) = \pi_E(v)$. Smoothness of $\hat{H}$ implies smoothness of H (proof left to reader). Thus H is a smooth bundle map $E \to F$ covering the identity, and linear on fibers. By definition, H is therefore a bundle homomorphism.

For the remainder of these notes, we use the notation (3) for the correspondence (2).

[1] "Determined by 0-jets" is a phrase invented just for these notes, in order to have a name by which to refer to this property in Proposition 1.8. For any integer $r \geq 0$, there is an object called the *r-jet of a section of E at a point*. We do not define general r-jets in these notes, but the 0-jet of a section $s \in \Gamma(E)$ at a point p is simply the value $s(p) \in E_p$. In a sense that can be made precise, an r-jet of a section s at p captures the "r^{th}-order information" of s at p.

Since we have a canonical isomorphism $\mathrm{Hom}(E,F) \to F \otimes E^*$, the section $\hat{H}$ above is canonically identified with a section of $F \otimes E^*$. If (1) is satisfied, then the action of L on a section s is achieved by pointwise tensor-algebra operations:

$$\begin{array}{ccccc} \mathrm{Hom}(E,F)_p \times E_p & \to & F_p \otimes E_p^* \otimes E_p & \to & F_p \\ (\hat{H}_p, s_p) & \mapsto & \hat{H}_p \otimes s_p & \mapsto & \langle \hat{H}_p, s_p \rangle, \end{array}$$

where the last map is contraction on the last two factors of $F_p \otimes E_p^* \otimes E_p$ (induced by the trilinear map $F_p \times E_p^* \times E_p \to F_p$, $(w, \alpha, v) \mapsto w\langle \alpha, v \rangle$). This is why we call L tensorial if (1) is satisfied.

We will show that for a linear map $\Gamma(E) \to \Gamma(F)$, the notions of $\mathcal{F}$-linearity, tensoriality, and having the property of being determined by 0-jets, are equivalent. We start with some lemmas we will need.

Lemma 1.2 *Let $p \in M$. There exists a chart (U, ϕ) of M such that $E|_U$ is trivial.*

Proof: Let (U_1, ϕ) be a chart of M with $p \in U_1$. Let V be an open neighborhood of p such that $E|_V$ is trivial. Let $U = U_1 \bigcap V$, $\phi = \phi_1|_U$. Then (U, ϕ) is a chart with the desired property. ■

Lemma 1.3 *Let $p \in M$, Let (U, ϕ) be a chart of M such that $E|_U$ is trivial, let $B \subset \mathbf{R}^n$ be an open ball with $\overline{B} \subset \phi(U)$, and let $V = \phi^{-1}(B)$. Suppose $s \in \Gamma(E)$ is a section supported in $\overline{V}$ (i.e. identically zero on the complement). Assume that $\kappa := \mathrm{rank}(E) > 0$. Then there exist a κ-tuple $\{t_\mu \in \Gamma(E)\}_{\mu=1}^{\kappa}$ and a κ-tuple $\{h^\mu \in C^\infty(M)\}_{\mu=1}^{\kappa}$ such that (i) $\{t_\mu|_V\}_1^\kappa$ is a basis of sections of $E|_V$, and (ii) $s = \sum_\mu h^\mu t_\mu$.*

The point of this lemma is to show that any $s \in \Gamma(E)$ can be expressed *globally* as a "linear" combination, with coefficients in $\mathcal{F}$, of *global* sections of E that restrict to a basis of sections of $E|_V$.

Proof of Lemma 1.3: Let $\{s_\mu\}_{\mu=1}^{\kappa}$ be a basis of sections of $E|_U$. Let $\rho : M \to \mathbf{R}$ be a smooth function such that $\rho \equiv 1$ on V and $\rho \equiv 0$ on $M \setminus U$.

Since $\{s_\mu\}$ is a basis of sections of $E|_U$, there exist unique smooth functions $f^\mu : U \to \mathbf{R}$ such that $s|_U = \sum_{\mu=1}^{\kappa} f^\mu s_\mu$. Since $\mathrm{supp}(s) \subset \overline{V} \subset U$, and the s_μ are linearly independent at each point, we have $\mathrm{supp}(f^\mu) \subset \overline{V}$ for each μ. For each μ the function $\rho|_U f^\mu$ is smooth and supported in $\overline{V} \subset U$, hence extends smoothly by 0 to a function $h^\mu : M \to \mathbf{R}$ (still supported in $\overline{V}$). Similarly, the section $\rho|_U s_\mu$ is smooth and supported in U, hence extends smoothly by 0 to a section t_μ of E, supported in U.

Let $\tilde{s} = \sum_{\mu=1}^{\kappa} h^\mu t_\mu$. Then for $p \in M \setminus V$, we have $\tilde{s}(p) = 0 = s(p)$, since s and the h^μ are supported in $\overline{V}$. For $p \in V$, we have $\rho(p) = 1$, implying $h^\mu(p) = f^\mu(p)$ and $t_\mu(p) = s_\mu(p)$, hence implying $\tilde{s}(p) = s(p)$.

Therefore, we have the global equalities $s = \tilde{s} = \sum h^\mu t_\mu$. ■

Lemma 1.4 *Let $L : \Gamma(E) \to \Gamma(F)$ be $\mathcal{F}$-linear. Let $p \in M$, $s \in \Gamma(E)$, and assume that $s(p) = 0$. Then $L(s)|_p = 0$.*

Proof: It suffices to assume that $\kappa := \mathrm{rank}(E) > 0$. Let (U, ϕ) be a chart of M such that $E|_U$ is trivial and $p \in U$. Let $B = B_r(\phi(p)) \subset \mathbf{R}^n$ be the open ball of radius r centered at $\phi(p) \in B$, with r small enough that and $\overline{B} \subset \phi(U)$. Let $B_1 = B_{r/2}(\phi(p))$, $V = \phi^{-1}(B)$, and $V_1 = \phi^{-1}(B_1)$. Let $\rho : M \to \mathbf{R}$ be a smooth function such that $\rho \equiv 1$ on V_1 and $\rho \equiv 0$ on $M \setminus V$. Let $s_1 = \rho s$ and $s_2 = (1 - \rho)s$; thus $s = s_1 + s_2$.

Since $\mathrm{supp}(s_1) \subset V$, Lemma 1.3 implies that we can write $s_1 = \sum_{\mu=1}^{\kappa} h^\mu t_\mu$ for some functions $h^\mu \in C^\infty(M)$ and some sections $t_\mu \in \Gamma(E)$ such that $\{t_\mu\}_1^\kappa$ is a basis of sections of $E|_V$. Observe that $s_1(p) = 0$. Since $\{t_\mu(p)\}$ is a basis of E_p, it follows that $h^\mu(p) = 0$ for each μ. The $\mathcal{F}$-linearity of L implies that $L(s_1) = \sum h^\mu L(t_\mu)$. Hence for all $p \in M$, $L(s_1)_p = \sum h^\mu(p) L(t_\mu)|_p = 0$.

Again using $\mathcal{F}$-linearity, $L(s_2)|_p = (1 - \rho(p))L(s)|_p = 0$, since $\rho(p) = 1$. Hence $L(s)|_p = L(s_1)|_p + L(s_2)|_p = 0$. ■

Corollary 1.5 *Let $L : \Gamma(E) \to \Gamma(F)$ be $\mathcal{F}$-linear. Then L is determined by 0-jets.*

Proof: Let $p \in M$, let $s_1, s_2 \in \Gamma(E)$, and assume that $s_1(p) = s_2(p)$. Let $s = s_2 - s_1$. Then $s(p) = 0$, so by Lemma 1.4, $L(s)|_p = 0$. Hence $L(s_2)|_p = L(s + s_1)|_p = L(s_1)|_p$. ■

Lemma 1.6 (Extendability of sections defined at a point) *Let $p \in M$, $v \in E_p$. There exists $s \in \Gamma(E)$ that extends v.*

Proof: Let V be an open neighborhood of p such that $E|_V$ is trivial, and let $\{s_\mu\}_1^\kappa$ be a basis of sections of $E|_V$. Let $\{c^\mu \in \mathbf{R}\}_{\mu=1}^\kappa$ be such that $v = \sum c^\mu s_\mu(p)$.

Let $\rho : M \to \mathbf{R}$ be a smooth function such that $\rho(p) = 1$ and $\mathrm{supp}(\rho) \subset V$. Define $s' \in \Gamma(E|_V)$ by $s' = \rho \sum c^\mu s_\mu$; thus $s'(p) = v$. Let s be the extension of s' by 0 to M. Then s is smooth, hence a section of E, and $s(p) = v$. ■

Corollary 1.7 *If $L : \Gamma(E) \to \Gamma(F)$ is tensorial, then the bundle homomorphism H in* (1) *is unique.*

Proof: Let H_1, H_2 be bundle homorphisms $E \to F$ satisfying (1). Let $H' = H_2 - H_1$ (defined pointwise). Then $(H_2 - H_1)(s(p)) = 0$ for all $p \in M, s \in \Gamma(E)$. Since for all $v \in E$, there exists a section $s \in \Gamma(E)$ extending v, it follows that $(H_2 - H_1)(v) = 0$ for all $v \in E$. Hence $H_2 = H_1$. ■

Proposition 1.8 *Let* $L : \Gamma(E) \to \Gamma(F)$ *be a linear map. Then the following are equivalent:*

(i) L *is* $\mathcal{F}$*-linear.*

(ii) L *is determined by* 0*-jets.*

(iii) L *is tensorial.*

Proof: We show "(iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)."

(iii) $\Rightarrow$ (i): This follows immediately from (1) and the definition of "bundle homomorphism covering the identity".

(i) $\Rightarrow$ (ii): This is Corollary 1.5.

(ii) $\Rightarrow$ (iii): Assume that L is determined by 0-jets. For $p \in M$ and $v \in E_p$, define $\hat{H}_p(v) \in F_p$ by

$$\hat{H}_p(v) := L(s)|_p, \tag{4}$$

where s is any extension of v to a section of E (such an extension exists by Lemma 1.6). Since L is determined by 0-jets, $\hat{H}_p(v)$ is well-defined; all extensions s of v yield the same value of $L(s)|_p$. Letting v vary over E_p, (4) therefore defines a map $\hat{H}_p : E_p \to F_p$.

If s_1, s_2 are extensions of $v_1, v_2 \in E_p$ to sections of E, and $c_1, c_2 \in \mathbf{R}$, then then $c_1 s_1 + c_2 s_2$ is an extension of $c_1 v_1 + c_2 v_2$ to a section of E. The linearity of L therefore implies that, for each p, the map $\hat{H}_p : E_p \to F_p$ is linear. Hence, letting p vary, we obtain a set-theoretic section $\hat{H}$ of $\mathrm{Hom}(E, F)$.

We next show that $\hat{H}$ is smooth. Let $p \in M$, let U be a neighborhood of p such that $E|_U$ and $F|_U$ are trivial, and let $\{s_\mu\}_{\mu=1}^{\kappa_1}, \{\sigma_\nu\}_{\nu=1}^{\kappa_2}$, be bases of sections of $E|_U, F|_U$ respectively. Let $\{\xi^\nu\}_{\nu=1}^{\kappa_2}$ be the basis of sections of $F^*|_U$ that is dual (pointwise) to $\{\sigma_\nu\}_{\nu=1}^{\kappa_2}$. Let $A : U \to \{\kappa_2 \times \kappa_1 \text{ matrices}\}$ be the function defined pointwise by expanding the elements $\hat{H}_q(s_\mu(q)) \in F_q$ in terms of the basis $\{\sigma_\nu(q)\}$ of F_q:

$$\hat{H}_q(s_\mu(q)) = \sum_{\nu=1}^{\kappa_2} \sigma_\nu(q) A^\nu{}_\mu(q), \quad q \in U, \quad 1 \le \mu \le \kappa_1 \ .$$

Alternatively, $A^\nu{}_\mu(q) = \langle \xi^\nu|_q, \hat{H}(s_\mu(q))\rangle$. To show that $\hat{H}$ is smooth at p, it suffices to show that the each coefficient-function $A^\nu{}_\mu$ is smooth at p.

Let $\rho : M \to \mathbf{R}$ be a smooth function such that $\rho \equiv 1$ on some open neighborhood V of p and $\rho \equiv 0$ on $M \setminus U$. The sections $\rho|_U s_\mu$ of $E|_U$ extend smoothly by 0 to sections t_μ of E, and we have $t_\mu(q) = s_\mu(q)$ for all $q \in V$. Hence for $q \in V$,

$$A^\nu{}_\mu(q) = \langle \xi^\nu|_q, \hat{H}(s_\mu(q))\rangle = \langle \xi^\nu|_q, \hat{H}(t_\mu(q))\rangle = \langle \xi^\nu|_q, L(t_\mu)|_q\rangle \tag{5}$$

Since $t_\mu \in \Gamma(E)$, $L(t_\mu)$ is a (smooth) section of F. Hence both ξ^ν and $L(t_\mu)$ are smooth on V, so (5) implies that the functions $A^\nu{}_\mu$ are smooth on V. In particular, they are smooth at p.

Thus $\hat{H}$ is smooth at p. Since p was arbitrary, $\hat{H} \in \Gamma(E)$. Using the correspondence (2)–(3), we obtain a bundle homomorphism $H : E \to F$ such that (1) holds. Hence L is tensorial. ■

Notation 1.9 For vector bundles E, F over M, let $\mathrm{Hom}_{\mathcal{F}}(\Gamma(E), \Gamma(F))$ denote the set of $\mathcal{F}$-linear maps $\Gamma(E) \to \Gamma(F)$.

Remark 1.10 For $L \in \mathrm{Hom}_{\mathcal{F}}(\Gamma(E), \Gamma(F))$ let us write H_L for the unique bundle homomorphism for which $L(s) = H_L \circ s$ (uniqueness being guaranteed by Corollary 1.7). Then, using (2)–(3), we obtain a natural map

$$\begin{aligned} j_F^E : \mathrm{Hom}_{\mathcal{F}}(\Gamma(E), \Gamma(F)) &\to \Gamma(\mathrm{Hom}(E, F)), \\ L &\mapsto \hat{H}_L \ . \end{aligned} \tag{6}$$

Observe that $\mathrm{Hom}_{\mathcal{F}}(\Gamma(E), \Gamma(F))$ is a vector space—a subspace of $\mathrm{Hom}(\Gamma(E), \Gamma(F))$ —and, by Proposition (1.8), is the same as the space of tensorial maps $\Gamma(E) \to \Gamma(F)$. Furthermore, the space $\mathrm{Hom}_{\mathcal{F}}(\Gamma(E), \Gamma(F))$ is itself an $\mathcal{F}$-module, and it is easily seen that j_F^E is an $\mathcal{F}$-module isomorphism.

Remark 1.11 A map $L : \Gamma(E) \to \Gamma(F)$ is called *local* if, for all $p \in M$ and $s \in \Gamma(E)$, the value of $L(s)|_p$ depends only on the germ of s at p. (If L is linear, then L is local if and only if for every open set $U \subset M$ and every $s \in \Gamma(E)$ such that $s|_U \equiv 0$, we have $L(s)|_V \equiv 0$.) Obviously, if L depends only on 0-jets, then L is local, but the converse is very far from true. Contained in the set of all local maps[2] $\Gamma(E) \to \Gamma(F)$ is the set of all *differential operators* $\Gamma(E) \to \Gamma(F)$. A *differential operator* $\Gamma(E) \to \Gamma(F)$ *of order 0* is simply a map that is determined by 0-jets. In these notes we do not define what "differential operator $\Gamma(E) \to \Gamma(F)$" means in general, but a true fact is that for each integer $r \geq 0$ there is a notion of *differential operator of order* r. As one might expect from the name "differential operator", and the local nature of anything that we generally call "differentiation", any differential operator of any order is local.

2 $\mathcal{F}$-multilinearity and tensoriality

For *any* vector spaces V, W, finite- or infinite-dimensional, we write $\mathrm{Hom}(V, W)$ for the space of all linear maps $V \to W$. In this notation, we do not care if the vector spaces are topologized, let alone whether our linear maps are continuous.

[2]The term "local map" is being used here with the very specific meaning above. Outside this context, the same term could be used to mean something very different, e.g. a map defined just on some open subset of a topological space.

Definition 2.1 Let $E_1, E_2, \ldots, E_r, F$ be vector bundles over M, and let $L : \Gamma(E_1) \times \Gamma(E_2) \times \cdots \times \Gamma(E_r) \to \Gamma(F)$ be a map.

1. We say that L is *$\mathcal{F}$-multilinear* if for $1 \le i \le r$, L is $\mathcal{F}$-linear as a function of its i^{th} argument with the other arguments held fixed.

2. We say that L is *tensorial* if there exists a bundle homomorphism $H : E_1 \otimes E_2 \ldots \otimes E_r \to F$, covering the identity, such that for all $s_i \in \Gamma(E_i)$, $1 \le i \le r$,
$$L(s_1, s_2, \ldots, s_r) = H \circ (s_1 \otimes s_2 \ldots \otimes s_r). \tag{7}$$
Here $s_1 \otimes s_2 \ldots \otimes s_r$ is the section of $E_1 \otimes E_2 \ldots \otimes E_r$ defined by pointwise tensor-product:
$$\begin{array}{rcl}(s_1 \otimes s_2 \ldots \otimes s_r)|_p = s_1(p) \otimes s_2(p) \ldots \otimes s_r(p) & \in & E_1|_p \otimes E_2|_p \ldots \otimes E_r|_p \\ & \underset{\text{canon.}}{\cong} & (E_1 \otimes E_2 \ldots \otimes E_r)_p \ .\end{array}$$

3. For $s_i \in \Gamma(E_i), 1 \le i \le r$, and $p \in M$, we say that $L(s_1, \ldots, s_r)|_p$ *depends only of values of* $s_1, \ldots, s_r$ *at* p if for all $s_i' \in \Gamma(E_i)$ with $s_i'(p) = s_i(p)$, $1 \le i \le r$, we have $L(s_1', \ldots, s_r')|_p = L(s_1, \ldots, s_r)|_p$. In these notes, we will say that L *is determined by* 0*-jets* if for all $s_i \in \Gamma(E_i), 1 \le i \le r$, and $p \in M$, the value $L(s_1, \ldots, s_r)|_p$ depends only of values of $s_1, \ldots, s_r$ at p.

Note that from (2)–(3), we have a natural one-to-one correspondence

$$\begin{array}{l}\{\text{bundle homomorphisms } E_1 \otimes E_2 \otimes \ldots \otimes E_r \to F\} \\ \qquad \longleftrightarrow \Gamma(\mathrm{Hom}(E_1 \otimes E_2 \otimes \ldots \otimes E_r, F)),\end{array} \tag{8}$$

which we will again denote by $H \longleftrightarrow \hat{H}$.

For any vector spaces V, W, Z, let $\mathrm{Bihom}(V \times W, Z)$ denote set of bilinear maps $V \times W \to Z$. This set is a vector subspace of $\mathrm{Maps}(V \times W, Z)$, the set of all functions $V \times W \to Z$.

Recall that for any nonempty *sets* X, Y, Z, the natural map

$$\begin{array}{rcl}\mathrm{Maps}(X \times Y, Z) & \stackrel{\natural}{\to} & \mathrm{Maps}(X, \mathrm{Maps}(Y, Z)), \\ f & \mapsto & f_\natural : x \mapsto \{y \mapsto f(x, y)\}\end{array}$$

is a one-to-one correspondence. For vector spaces V, W, Z one can easily verify that $\natural$ restricts to an isomorphism

$$\natural_{\mathbf{R}} : \mathrm{Bihom}(V \times W, Z) \to \mathrm{Hom}(V, \mathrm{Hom}(W, Z))$$

Proposition 2.2 *Let E_1, E_2, F be vector bundles over M and let $L : \Gamma(E_1)\times\Gamma(E_2) \to \Gamma(F)$ be a bilinear map. Then the following are equivalent:*

(i) L is $\mathcal{F}$-bilinear.

(ii) L is determined by 0-jets.

(iii) L is tensorial.

Proof: We show "(iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)."

(iii) $\Rightarrow$ (i): This follows immediately from (7).

(i) $\Rightarrow$ (ii): Assume L is $\mathcal{F}$-bilinear. Then for fixed $s_1 \in \Gamma(E_1)$, the map $\Gamma(E_2) \to \Gamma(\mathcal{F})$, $s_2 \mapsto L(s_1, s_2)$, is $\mathcal{F}$-linear. Hence by Corollary 1.5, holding s_1 fixed, for $p \in M$ the value $L(s_1, s_2)|_p$ depends only on the value of s_2 at p. Similarly, with s_2 held fixed, the value $L(s_1, s_2)|_p$ depends only on the value of s_1 at p. Hence given $p \in M$, and sections s_1, s_1' of E_1, s_2, s_2' of E_2, such that $s_i(p) = s_i'(p)$ for $i = 1, 2$, we have $L(s_1', s_2')|_p = L(s_1', s_2)|_p = L(s_1, s_2)|_p$. Hence $L(s_1, s_2)|_p$ depends only on the values of s_1 and s_2 at p. Thus L is determined by 0-jets.

(ii) $\Rightarrow$ (iii): Assume that L is determined by 0-jets. Let $L' = \natural_{\mathbf{R}}(L) \in \mathrm{Hom}(\Gamma(E_1), \mathrm{Hom}(\Gamma(E_2), \Gamma(F)))$. Fix $s_1 \in \Gamma(E_1)$. Then $L'(s_1) \in \mathrm{Hom}(\Gamma(E_2), \Gamma(F))$ depends only on 0-jets. Hence, by Proposition 1.8, $L'(s_1)$ is $\mathcal{F}$-linear and tensorial, so there exists $\widehat{H}^{(s_1)} \in \Gamma(\mathrm{Hom}(E_2, F))$ such that for all $p \in M$, $L(s_1, s_2)|_p = (L'(s_1))(s_2)\big|_p = \widehat{H}^{(s_1)}|_p(s_2(p))$.

Letting s_1 vary, we now have a map $L'' : \Gamma(E_1) \to \Gamma(\mathrm{Hom}(E_2, F))$, $s_1 \mapsto \widehat{H}^{(s_1)}$. Since L is linear, so is L'', and since L is determined by 0-jets, so is L''. Hence, using Proposition 1.8, again L'' is tensorial, so there exists $\widehat{H} \in \Gamma(\mathrm{Hom}(E_1, \mathrm{Hom}(E_2, F)))$ such that for all $p \in M$, $L(s_1, s_2)|_p = (L'(s_1))(s_2)\big|_p = \left(\widehat{H}_p(s_1(p))\right)(s_2(p))$. But using the canonical isomorphisms

$$\begin{aligned}
\mathrm{Hom}(E_1, \mathrm{Hom}(E_2, F)) &\underset{\text{canon.}}{\cong} \mathrm{Hom}(E_2, F) \otimes E_1^* \\
&\underset{\text{canon.}}{\cong} F \otimes E_2^* \otimes E_1^* \\
&\underset{\text{canon.}}{\cong} F \otimes E_1^* \otimes E_2^* \\
&\underset{\text{canon.}}{\cong} F \otimes (E_1 \otimes E_2)^* \\
&\underset{\text{canon.}}{\cong} \mathrm{Hom}(E_1 \otimes E_2, F),
\end{aligned}$$

we can canonically identify $\hat{H}$ with a section of $\mathrm{Hom}(E_1 \otimes E_2, F)$. It follows from the correspondence (8) that L is tensorial. ■

Corollary 2.3 *Let $E_1, E_2, \ldots, E_r, F$ be vector bundles over M and let $L : \Gamma(E_1) \times \Gamma(E_2) \times \cdots \times \Gamma(E_r) \to \Gamma(F)$ be a multilinear map. Then the following are equivalent:*

(i) L is $\mathcal{F}$-multilinear.

(ii) L is determined by 0-jets.

(iii) L is tensorial.

Proof: Proposition 2.2 plus induction (more or less). ■