Differential Geometry—MTG 6257—Spring 2018
Problem Set 1
Due-date: Wednesday, 2/7/18

Required problems (to be handed in): 2bd, 3a, 5, and 6cef. In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Required reading: (1) The first two paragraphs of problem 7, and Remarks 1 and 2 at the end of problem 7. (2) The statements of *results* (Propositions, Corollaries, Lemmas, Theorems), and Remark 0.12, in the handout, "Some Sufficient Conditions for Paracompactness". The terminology "σ-compact" and "exhaustion" are defined on p. 1.

Optional reading: The *proofs* in the paracompactness handout.

Optional problems: All the ones that are not required. If you are interested in de Rham cohomology, problems 8 and 9 are fundamental. The non-technical parts are worth reading, even if you don't do the problems. (The technical parts are just in problem 8: the set-up and part (a).)

1. Let M be an oriented n-dimensional manifold. Show that the map $\Omega_c^n(M) \to \mathbf{R}$ given by $\omega \mapsto \int_M \omega$ is linear.

2. Let M, N be manifolds of the same dimension n. If M and N are oriented, then at each $p \in M$, a local diffeomorphism $F : M \to N$ either *preserves* orientation or *reverses* it, accordingly as F_{*p} carries a positively-oriented basis of T_pM to a positively- or negatively-oriented basis of $T_{F(p)}N$. We say that F is an *orientation-preserving map* if it preserves orientation at each point, and an orientation-reversiing map if it reverses orientation at each point. For such maps, below we define $\mathrm{sgn}(F) = 1$ (respectively, -1) if F preserves (respectively, reverses) orientation.

(a) Show that if M is connected, and M and N are oriented, then every local diffeomorphism is either orientation-preserving or orientation-reversing.

(b) Show that if M is connected $F : M \to N$ is a diffeomorphism, then for every $\omega \in \Omega_c^n(N)$,

$$\int_M F^*\omega = \mathrm{sgn}(F) \int_N \omega.$$

(This fact is called *invariance of the integral under diffeomorphism.*)

(c) Show that if N is oriented, every local diffeomorphism $F : M \to N$ canonically induces an orientation on M, which may be characterized as the unique orientation on M such that F is orientation-preserving. (Note that this implies that if N is orientable, then so is M.)

(d) Suppose that M is compact, N is connected, and $F : M \to N$ is a local diffeomorphism (hence a submersion, for dimensional reasons). From last semester's homework, these conditions imply that F is also a covering map. Connectedness of N implies that the cardinality of the set $F^{-1}(\{p\})$ is independent of the point $p \in N$ (a basic fact about covering spaces), and compactness of M implies that this cardinality is finite.

Assume that N is oriented and that M is given the induced orientation as in part (c). Let $\deg(F)$, the *degree* of the covering map, denote the cardinality of $F^{-1}(\{p\})$ for some (hence any) p. Show that for all $\omega \in \Omega^n(N)$,

$$\int_M F^*\omega = \deg(F) \int_N \omega.$$

3. **Extension from a closed submanifold.** This problem is another valuable application of partitions of unity. You should find the arguments for all three parts very similar to each other.

Let M be a manifold, $Z \subset M$ a submanifold that is closed as a subset of M.

(a) "*Smooth Tietze Extension Theorem*". Suppose $f : Z \to \mathbf{R}$ is a smooth function. Show that f can be extended to a smooth function $M \to \mathbf{R}$.

Note: This would be false without the hypothesis that Z is closed in M, even if we were looking just for *continuous* extensions, and even if we required $\dim(Z)$ to be strictly smaller than $\dim(M)$. If your argument doesn't use the hypothesis that Z is closed, you've made a mistake. The same goes for parts (b) and (c).

(b) A *vector field along* Z is a section of $TM|_Z$, i.e. a smooth map $X : Z \to TM$, $p \mapsto X_p \in T_pM$. (We do not require X_p to be tangent to Z.) Show that a vector field along Z can be extended to a vector field on M.

(c) Similarly, for $k > 0$ a *k-form along* Z is a map $\omega : Z \to \bigwedge^k T^*M$, $p \mapsto \omega_p \in \bigwedge^k T_p^*M$, smooth in the sense that if $X_1, \ldots X_k$ are smooth vector fields along Z, then $p \mapsto \omega(X_1, \ldots, X_k)\big|_p$ is smooth. Show that a k-form along Z can be extended to a k-form on M.

4. Let D be a domain with regular boundary in an oriented n-dimensional manifold M, where $n \geq 1$ and let ∂D have the induced orientation. Let $\omega \in \Omega^p(M), \eta \in \Omega^q(M)$, where $p + q = n - 1$, and assume that at least one of the sets $\mathrm{supp}(\omega) \bigcap \overline{D}$, $\mathrm{supp}(\eta) \bigcap \overline{D}$, is compact. (Note that the compact-support assumption is superfluous if we assume that M is compact or that $\overline{D}$ is compact.) Prove the "integration-by-parts" formula

$$\int_D d\omega \wedge \eta = \int_{\partial D} \omega \wedge \eta - (-1)^p \int_D \omega \wedge d\eta.$$

Remark. The case $D = M$ (hence $\partial D = \emptyset$) is important all by itself.

5. Let M be an n-dimensional oriented manifold. A *volume form* on M is a positive n-form. (We showed in class that a volume form always exists.) Show that if M is compact and ω is a volume form, then ω is not exact.

6. Let M be an n-dimensional manifold, $n \geq 1$. The *orientation double-cover* $\widetilde{M}$ is a covering manifold of M that can be constructed as follows. For each $p \in M$ let $\mathrm{Orn}(p)$ denote the set of orientations of T_pM, a two-element set. Given $\sigma \in \mathrm{Orn}(p)$, we let $-\sigma$ denote the other orientation. As a set, let $\widetilde{M} = \bigcup_{p\in M} \mathrm{Orn}(p)$. There is a natural two-to-one map $\pi : \widetilde{M} \to M$ carrying both elements of $\mathrm{Orn}(p)$ to p. We give $\widetilde{M}$ the topology induced by the map π (i.e. a set $\tilde{U} \subset \widetilde{M}$ is declared to be open if and only if $\pi(\tilde{U})$ is open).

It can be shown that every manifold has as an atlas $\{(U_\alpha, \phi_\alpha)\}$ for which all the sets U_α and intersections $U_\alpha \bigcap U_\beta$ are connected[1]; let $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ be such an atlas for M. Then, for each $\alpha \in A$, the set $\pi^{-1}(U_\alpha)$ has two connected components, which are distinguished from each other as follows. For $p \in U_\alpha$ let $\sigma_\alpha(p)$ be the orientation of T_pM pulled back by the map $\phi_\alpha : U_\alpha \to \mathbf{R}^n$, where $\mathbf{R}^n$ has its standard orientation. Each $\tilde{p} \in \pi^{-1}(U_\alpha)$ is, by definition, an orientation of $T_{\pi(p)}M$; hence $\tilde{p} = \pm\sigma_\alpha(\pi(\tilde{p}))$ (where "$+\sigma$" means σ). The sign in this formula is constant on each connected component of $\pi^{-1}(U_\alpha)$. We define $\tilde{U}_{\alpha,+}$ to be the component on which $\tilde{p} = \sigma_\alpha(\pi(\tilde{p}))$, and $\tilde{U}_{\alpha,-}$ to be the component on which $\tilde{p} = -\sigma_\alpha(\pi(\tilde{p}))$. We define corresponding chart-maps $\tilde{\phi}_{\alpha,\pm} : \tilde{U}_{\alpha,\pm} \to \mathbf{R}^n$ as follows. Let $r : \mathbf{R}^n \to \mathbf{R}^n$ be the reflection $(x^1, x^2, \ldots x^n) \mapsto (-x^1, x^2, \ldots, x^n)$. Then we define $\tilde{\phi}_{\alpha,+} = \phi_\alpha \circ \pi$, $\tilde{\phi}_{\alpha,-} = r \circ \phi_\alpha \circ \pi$.

(a) Let $\tilde{A} = A \times \{+,-\}$, an index set for the pairs $(\tilde{U}_{\alpha,\pm}, \tilde{\phi}_{\alpha,\pm})$ constructed above. Show that $\{\tilde{U}_{\tilde{\alpha}}, \tilde{\phi}_{\tilde{\alpha}}\}_{\tilde{\alpha}\in\tilde{A}}$ is an atlas for $\widetilde{M}$, hence that $\widetilde{M}$ is a manifold. (You may assume that paracompactness and Hausdorffness of M imply that $\tilde{M}$ has these properties.)

(b) If M is connected, what is the relation between connectedness of $\widetilde{M}$ and orientability of M? (I'm not asking for a proof here; I just want you to state what the relation is.)

(c) Show that the atlas $\{\tilde{U}_{\tilde{\alpha}}, \tilde{\phi}_{\tilde{\alpha}}\}_{\tilde{\alpha}\in\tilde{A}}$ is oriented. Hence $\widetilde{M}$ is orientable; even better, the construction above gives it a *canonical orientation*, the one induced by this atlas. (It can be shown that this orientation is independent of the atlas of M that we started with, but I'm not asking you to show that.)

(d) Show that $\pi : \widetilde{M} \to M$ is a submersion and a smooth, two-to-one covering map.

(e) Since every point in $\widetilde{M}$ is an orientation of a vector space, there is a natural map $\tau : \widetilde{M} \to \widetilde{M}$ defined by $\tau(\sigma) = -\sigma$ (this map is called an *involution*, a term you

[1] It takes some work to show this; just assume it's true for purposes of this problem.

may recall from group theory, because $\tau \circ \tau$ is the identity map). Show that τ is an orientation-reversing map.

(f) Since $\widetilde{M}$ is oriented, we may integrate any compactly supported n-form over $\widetilde{M}$. Show that if $\omega \in \Omega^n(M)$ is compactly supported, then so is $\pi^*\omega$, and

$$\int_{\widetilde{M}} \pi^*\omega = 0.$$

Hint for doing this quickly and elegantly: part (e).

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Some notation used below. For any manifold M, point $p \in M$, tangent vector $X \in T_pM$, and $\omega \in \bigwedge^k T_p^*M$ $(k \geq 1)$ we define $\iota_X\omega \in \bigwedge^{k-1} T_p^*M$ by

$$(\iota_X\omega)(Y_1, \dots Y_{k-1}) = \omega(X, Y_1, \dots, Y_{k-1}) \quad \forall Y_1, \dots Y_{k-1} \in T_pM.$$

Given any vector field X on M and any $\omega \in \Omega^k(M)$, a $(k-1)$-form $\iota_X\omega$ is defined by applying the above operation pointwise. For the sake of completeness, if $k = 0$ we define $\iota_X\omega = 0$.

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7. *"Explicit" Poincaré Lemma for star-shaped regions.* The classical Poincaré Lemma asserts that, for all n and all $k > 0$, every closed k-form on $\mathbf{R}^n$ is exact.

Recall that a set U in a vector space is *star-shaped* if there exists $p \in U$ such that for all $q \in U$, the line segment from p to q lies entirely in U. Given such p, we may say that U is "star-shaped with respect to p".[2] In particular, $\mathbf{R}^n$ is star-shaped. In this problem we establish that if U is an open star-shaped subset of $\mathbf{R}^n$, then every closed k-form on U $(k > 0)$ is exact. (Thus the Poincaré Lemma follows as a special case.) There are many ways of showing this; the point of this problem is to give an explicit formula that produces, for each closed form $\omega \in \Omega^k(U)$, a form $\eta \in \Omega^{k-1}(U)$ such that $\omega = d\eta$.

It suffices to produce such a formula under the hypothesis that U is star-shaped with respect to the origin, which we henceforth assume; a more general formula can be obtained from this by applying a translation. The case $n = 0$ is trivial, so we also assume $n > 0$.

Set-up. For $t \in [0, 1]$ define $F_t : \mathbf{R}^n \to \mathbf{R}^n$ by $F_t(x) = tx$. Since U is star-shaped with respect to the origin, $F_t(U) \subset U$. Let V be the vector field $\sum_i x^i \frac{\partial}{\partial x^i}$. For $k > 0$ and $\omega \in \Omega^k(U)$, define $P(\omega) \in \Omega^{k-1}(U)$ by

$$P(\omega) = \int_0^1 t^{-1} F_t^*(\iota_V\omega)\, dt,$$

interpreted pointwise:

[2]The set U is *convex* if it is star-shaped with respect to each of its points.

$$P(\omega)\big|_x = \int_0^1 t^{-1}\left(F_t^*(\iota_V\omega)\right)\big|_x\, dt. \tag{1.1}$$

Despite appearances, this integral is not improper: if we write ω as $\sum_I f_I\, dx^I$, where the sum is over increasing multi-indices of length k, then $\left(F_t^*(\iota_V\omega)\right)\big|_x = \sum_I t^k f_I(tx)\, \iota_{V_x} dx^I$, so the integrand in (1.1) is $O(t^{k-1})$ as $t \to 0$. **(End of set-up.)**

Your job: Show that if ω is closed, then $\omega = d(P(\omega))$ (and hence that ω is exact).

Remark 1. With $U = \mathbf{R}^3$, we have seen that there is a dictionary translating between "curl of a vector field" (interpreting "vector field" as in Calc 3) and "d of a 1-form", and "between divergence of a vector field" and "d of a 2-form". Given a vector field X such that $\nabla\cdot X = 0$, the map P above (with $k = 2$) provides one way to construct a vector field A such that $X = \nabla \times A$.[3]

Remark 2. As seen in class, for any connected manifold M we have $H^0_{\mathrm{DR}}(M) = \mathbf{R}$. Hence the Poincaré Lemma, generalized to star-shaped regions U as above, can be written as

$$H^k_{\mathrm{DR}}(U) \cong \begin{cases} \mathbf{R} & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases} \tag{1.2}$$

More generally, (1.2) holds under the much weaker assumption that U is *contractible* (see problem 9), but it is harder to write down an explicit formula analogous to "$P(\omega)$" in that generality.

The remaining two problems, also about de Rham cohomology, are inspired by the presentation in Bott and Tu, *Differential Forms in Algebraic Topology.*

8. This problem gives a proof that, for any manifold M and any $k \geq 0$, $H^k_{\mathrm{DR}}(M \times \mathbf{R}) \cong H^k_{\mathrm{DR}}(M)$. This fact, plus induction, plus the trivial fact that (1.2) holds for $U = \mathbf{R}^0$, yield another proof of the Poincaré Lemma.

Below, we simply write "H^k" for "H^k_{DR}".

Set-up. Fix a manifold M. For all $(p,t) \in M \times \mathbf{R}$, recall that we can *canonically* identify $T_{(p,t)}(M \times \mathbf{R})$ as $T_pM \oplus T_t\mathbf{R}$. There is a similar canonical identification of cotangent spaces. Hence, letting t denote the standard coordinate on $\mathbf{R}$, there is a well-defined vector field on $M \times \mathbf{R}$ whose value at (p, t_0) is $(0_{T_pM}, \frac{\partial}{\partial t}\big|_{t_0})$, which (with a slight abuse of notation) we will denote $\frac{\partial}{\partial t}$. Similarly, we have a well-defined 1-form dt on $M \times \mathbf{R}$.

For $k \geq 1$, $\omega \in \Omega^k(M)$, and $(p,t) \in M \times \mathbf{R}$, the value of ω at (p,t) can be written uniquely as $\omega'(p,t) + dt \wedge \omega''(p,t)$, where $\omega'(p,t) \in \bigwedge^k T_p^*M$ and $\omega''(p,t) \in \bigwedge^{k-1} T_p^*M$.

[3] In case you know the relevant physics: this construction of a vector potential is not terribly useful for E&M, since the regions in which we want to find vector potentials for the magnetic field, e.g. $\mathbf{R}^3$ with a curve removed (for a wire carrying current) are *never* star-shaped.

(This decomposition may also be characterized by:

$$\omega'(p,t) = s_t^*\left(\iota_{\partial/\partial t}(dt \wedge \omega(p,t))\right), \quad \omega''(p,t) = s_t^*\left(\iota_{\partial/\partial t}\,\omega(p,t))\right), \tag{1.3}$$

where $s_t : M \to M \times \mathbf{R}$ is the map $p \mapsto (p,t)$. You may wish to convince yourself of this by introducing local coordinates $\{x^i\}$ on M. We can then write $\omega(p,t)$ as $\sum_{|I|=k} a_I\, dx^I + \sum_{|J|=k-1} b_J\, dt \wedge dx^J$, where the sums are over increasing multi-indices of the indicated lengths, and where if $k=1$, we interpret the sum over J just as $b\,dt$ for some real number b. Then $\omega'(p,t) = \sum_{|I|=k} a_I\, dx^I$ and $\omega'' = \sum_{|J|=k-1} b_J\, dx^J$, which can be recovered from the coordinate-independent characterization (1.3).)

For each p, the map $t \mapsto \omega''(p,t)$ is a continuous (in fact smooth) function $\mathbf{R} \to \bigwedge^{k-1} T_p^* M$. Hence we can define a linear map $S : \Omega^k(M \times \mathbf{R}) \to \Omega^{k-1}(M \times \mathbf{R})$ by

$$S(\omega)\big|_{(p,t)} = \int_0^t \omega''(p,s)\, ds;$$

for each p the right-hand side is an ordinary Riemann integral of a continuous vector-valued function. For $k=0$, we simply define $S(\omega)=0$. (**End of set-up**.)

(a) Make sense out of the following formula and show that it is true:

$$d\omega = d_M\omega' + dt \wedge \left(\frac{\partial \omega'}{\partial t} - d_M\omega''\right).$$

(b) Show that for all $\omega \in \Omega^k(M)$,

$$d(S(\omega)) + S(d\omega) = \omega - \pi^* s_0^*\, \omega, \tag{1.4}$$

where $\pi : M \times \mathbf{R} \to M$ is projection onto the first factor, and $s_0 : M \to M \times \mathbf{R}$ is the map $p \mapsto (p,0)$. Consequently, if ω is closed, then $\omega - \pi^* s_0^*\, \omega$ is exact.[4]

(c) Recall that if $F : N_1 \to N_2$ is a map of manifolds, we have $F^*(d\mu) = d(F^*\mu)$ for all differential forms μ on N_2. This implies that, for all $k \geq 0$, the linear map $F^* : \Omega^k(N_1) \to \Omega^k(N_1)$ carries closed forms to closed forms, and exact forms to exact forms, and therefore induces a linear map $H^k(N_2) \to H^k(N_1)$. It is common to denote this map also as F^*, but for clarity in this problem we will denote it as $F^\sharp$.

Show that the "chain rule for pullbacks", $(F \circ G)^* = G^* \circ F^*$, implies that for maps F, G that are composable as indicated, we have $(F \circ G)^\sharp = G^\sharp \circ F^\sharp$. Show also that if $F : N \to N$ is the identity map, then $F^\sharp : H^k(N) \to H^k(N)$ is also the identity (for all k).

(d) Letting I denote the identity map $H^k(M \times \mathbf{R}) \to H^k(M \times R)$, use parts (b) and (c) to show that the map $I - \pi^\sharp \circ s_0^\sharp = 0$ (the zero linear map), and hence that $\pi^\sharp \circ s_0^\sharp = I$.

[4] Students who've taken algebraic topology will recognize (1.4) as saying that S is a cochain homotopy, between the identity map and the map $\pi^* s_0^*$, on the cochain complex $\Omega^*(M \times \mathbf{R})$.

(e) Observing that $\pi \circ s_0$ is the identity map of M, show that $s_0^\sharp \circ \pi^\sharp$ is the identity map $H^k(M) \to H^k(M)$. Combining this with part (d), deduce that *the maps* $\pi^\sharp : H^k(M) \to H^k(M \times \mathbf{R})$ *and* $s_0^\sharp : H^k(M \times \mathbf{R}) \to H^k(M)$ *are isomorphisms, and are inverse to each other.*

(f) Show that $H^0(M \times \mathbf{R}) \cong H^0(M)$. Combining this with part (e), we therefore have

$$H^k(M \times \mathbf{R}) \cong H^k(M). \tag{1.5}$$

9. Let M, N be manifolds, and for $t \in \mathbf{R}$ define $s_t : M \to M \times \mathbf{R}$ by $s_t(p) = (p, t)$. Again let $\pi : M \times \mathbf{R} \to M$ be projection onto the first factor. From problem 8, for each $k \geq 0$ the maps $s_0^\sharp : H^k(M \times \mathbf{R}) \to H^k(M)$ and $\pi^\sharp : H^k(M) \to H^k(M \times \mathbf{R})$ are isomorphisms and are inverse to each other. Similarly, for any $t \in \mathbf{R}$ the map $s_t^\sharp : H^k(M \times \mathbf{R}) \to H^k(M)$ is an isomorphism that inverts $\pi^\sharp$.

Suppose that $F_0, F_1 : M \to N$ are smoothly homotopic maps, i.e. that there exists a smooth map $F : M \times [0,1] \to N$ such that $F_0 = F \circ s_0$ and $F_1 = F \circ s_1$. ($M \times [0,1]$ is a manifold-with-boundary; we define "smooth map" from a manifold-with-boundary to a manifold just as we did for maps from a manifold to a manifold.) Let $h : \mathbf{R} \to [0,1]$ be a smooth, monotone function such that $h(t) = 0$ for $t \leq 0$ and $h(t) = 1$ for $t \geq 1$; we saw in the "Bump Function" notes that such functions exist. Define $\tilde{F} : M \times \mathbf{R} \to N$ by $\tilde{F}(p,t) = F(p, h(t))$. Then $\tilde{F}$ is smooth, and its restriction to $M \times [0,1]$ is simply a reparametrization of the homotopy F. The purpose of introducing $\tilde{F}$ is just to put us in the realm where problem 8 applies directly.

(a) Show that for each $k \geq 0$ we have $F_0^\sharp = F_1^\sharp$ as maps $H^k(N) \to H^k(M)$.

(b) Let $G : M \to N$ be a constant map ($G(M) = \{\text{point}\}$). Show that for $k > 0$, $G^\sharp : H^k(N) \to H^k(M)$ is the zero map, and that for $k = 0$ the map $G^\sharp$ is injective.

(c) M is *smoothly contractible* if the identity map $M \to M$ is smoothly homotopic to a constant map. Show that if M is smoothly contractible, then

$$H^k_{\mathrm{DR}}(M) \cong \begin{cases} \mathbf{R} & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases} \tag{1.6}$$

(This yields yet another proof of the Poincaré Lemma. The map $\mathbf{R}^n \times [0,1] \to \mathbf{R}^n, (x,t) \mapsto tx$, is a homotopy between a constant map and the identity, so $\mathbf{R}^n$ is smoothly contractible.)

Remark: A true fact beyond the scope of this course (because of subject matter, not difficulty) is that if two smooth maps are homotopic, then they are smoothly homotopic. With this fact established, the word "smoothly" can be removed from "smoothly homotopic" and "smoothly contractible" in the above problem.