Differential Geometry—MTG 6257—Spring 2018
Problem Set 2
Due-date: Friday, 3/2/18

Required problems (to be handed in): 2bce, 3a, 4, 8a, 8b(i).

In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Optional problems: All the ones that are not required.

Required reading: All of the optional problems.

1. Let X be a vector field on a manifold M, with flow Φ, and let ω be a differential form on M or arbitrary degree. Just as we defined the Lie derivative (by X) of vector fields, 0-forms, and 1-forms, we define the Lie derivative of ω by X by

$$\mathcal{L}_X\omega = \frac{d}{dt}\left(\Phi_t^*\omega\right)\big|_{t=0}, \tag{1.1}$$

where (1.1) is interpreted pointwise: for each $p \in M$,

$$(\mathcal{L}_X\omega)\big|_p = \frac{d}{dt}\left((\Phi_t^*\omega)\big|_p\right)\Big|_{t=0}.$$

Show that $\mathcal{L}_X$ is Leibnizian with respect to wedge product:

$$\mathcal{L}_X(\omega\wedge\eta) = (\mathcal{L}_X\omega)\wedge\eta + \omega\wedge\mathcal{L}_X\eta$$

for all differential forms ω, η on M. (Note that, in contrast to the formula for $d(\omega\wedge\eta)$, there is no "$(-1)^{\deg(\omega)}$" in front of the second term this formula.)

Remark. Clearly $\omega \mapsto \mathcal{L}_X\omega$ is also linear. As mentioned last semester, a Leibnizian linear function on an algebra is also called a *derivation*. Thus $\mathcal{L}_X$ is a derivation on the exterior algebra $\Omega^\star(M) = \oplus_{k\geq 0}\Omega^k(M)$.

2. Let M be a manifold. Below, "vector field" and "differential form" mean "vector field on M " and "differential form on M", respectivly.

(a) Let X and Y be vector fields and let ω be a differential form of degree at least two. Find a simple relation between $\iota_X(\iota_Y\omega)$ and $\iota_Y(\iota_X\omega)$.[1]

(b) Let X be a vector field and let ω, η be differential forms.

$$\iota_X(\omega\wedge\eta) = (\iota_X\omega)\wedge\eta + (-1)^{\deg(\omega)}\omega\wedge\iota_X\eta.$$

[1]If we set $\Omega^k(M) = \{0\}$ for $k < 0$, the same relation holds without any restriction on the degree of ω.

Thus, like exterior derivative, ι_X is a *signed derivation* (also called *graded derivation* and *antiderivation* [not to be confused with "antiderivative", which is an entirely unrelated notion]) of the **Z**-graded algebra $\Omega^*(M)$. However, d increases degree by 1, while ι_X decreases degree by 1.

(c) Show that for $k \geq 1$, all k-forms ω, and all vector fields $X, Y_1, \ldots, Y_k$,

$$\begin{aligned} X\left(\omega(Y_1, Y_2, \ldots Y_k)\right) &= (\mathcal{L}_X\omega)(Y_1, ..., Y_k) + \omega(\mathcal{L}_X Y_1, Y_2, \ldots, Y_k) \\ &\quad + \omega(Y_1, \mathcal{L}_X Y_2, \ldots, Y_k) + \cdots + \omega(Y_1, Y_2, \ldots, \mathcal{L}_X Y_k). \end{aligned} \tag{1.2}$$

(d) Use the result of part (c) to show that Lie derivative by a vector field X is "Leibnizian with respect to contraction": for $k \geq 0$, all k-forms ω, and all vector fields X, Y,

$$\mathcal{L}_X(\iota_Y\omega) = \iota_{\mathcal{L}_X Y}\omega + \iota_Y \mathcal{L}_X\omega.$$

(e) Show that for any vector field X and any differential form ω,

$$\iota_X d\omega + d(\iota_X\omega) = \mathcal{L}_X\omega. \tag{1.3}$$

Note that this implies that if ω is *closed*, then $\mathcal{L}_X\omega$ is *exact*.[2] If ω is an n-form, where $n = \dim(M)$, then then trivially $d\omega = 0$, and (1.3) shows that the n-form $d(\iota_X\omega)$ that arose in our discussion of the Divergence Theorem is exactly $\mathcal{L}_X\omega$.

(f) Show that d commutes with Lie derivative by any vector field: for all vector fields X and differential forms ω,

$$d(\mathcal{L}_X\omega) = \mathcal{L}_X d\omega.$$

(There are at least two ways to do this: using part (e), or working directly from the definition (1.1).)

3. Let X, Y, Z be vector fields on a manifold M. Without introducing local coordinates, show that (1.2) and (1.3) imply the following.[3]

[2] Students who've taken algebraic topology have seen this argument before. Equation (1.3) implies that the map $\mathcal{L}_X : \Omega^*(M) \to \Omega^*(M)$ is (co-)chain homotopic to the zero map, and hence induces the zero map on de Rham cohomology—which is exactly the same as saying that $\mathcal{L}_X$ maps closed forms to exact forms.

[3] It's okay if you used local coordinates to establish formula (1.3). Just don't make any explicit use of local coordinates in problem 3.

(a) For $\omega \in \Omega^1(M)$.

$$d\omega(X,Y) = X(\langle \omega, Y\rangle) - Y(\langle \omega, X\rangle) - \langle \omega, [X,Y]\rangle.$$

(b) For $\omega \in \Omega^2(M)$,

$$\begin{aligned} d\omega(X,Y,Z) &= X\left(\omega(Y,Z)\right) - Y\left(\omega(X,Z)\right) + Z\left(\omega(X,Y)\right) \\ &\quad -\omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X). \end{aligned}$$

Hint: for any k-form ω and vector fields $X, Y_1, \ldots, Y_k$, we have $d\omega(X, Y_1, \ldots, Y_k) = (\iota_X d\omega)(Y_1, \ldots, Y_k)$.

4. Let M be a manifold and $Z \subset M$ a submanifold. Let X and Y be vector fields on Z, and assume that $U \subset M$ is an open set such that $X|_U$ and $Y|_U$ extend to (smooth) vector fields $\tilde{X}, \tilde{Y}$ on U. Show that for each $p \in Z \bigcap U$,

$$[X,Y]_p = [\tilde{X}, \tilde{Y}]_p.$$

(Here, as often before, we have identified T_pZ with $j_{*p}(T_pZ) \subset T_pM$, where $j: Z \to M$ is the inclusion map.) In particular, $[\tilde{X}, \tilde{Y}]_p$ is tangent to Z and is independent of the choices of extensions $\tilde{X}, \tilde{Y}$.

This can be done in *one or two sentences*, and *without any use of local coordinates*, using a homework problem from last semester.

5. Let E be a vector bundle over a manifold M. We define *germs of sections of* E just as we defined germs of real-valued functions:

1. For each $p \in M$, let $\tilde{\mathcal{G}}_p(E) = \{(U,s) \mid U$ is an open neighborhood of p, and $s \in \Gamma(E|_U)\}$.
2. Define an equivalence relation $\sim$ on $\tilde{\mathcal{G}}_p(E)$ by declaring $(U,s) \sim (U', s')$ if there exists some open set $V \subset U \cap U'$ containing p such that $s|_V = s'|_V$.
3. As a set, define $\mathcal{G}_p(E) = \tilde{\mathcal{G}}_p(E)/\sim$.

We then show, by the same argument as for real-valued germs, that the set $\mathcal{G}_p(E)$ inherits a natural vector-space structure from this construction. (You are not being asked to show this.)

Let $p \in M$. Show that every germ $[(U,s)]$ has a representative defined on all of M (i.e. is the germ, at p, of a (global) section of E).

6. Let M be a manifold, $Z \subset M$ a submanifold, and E a vector bundle over M. It is easily seen that $E|_Z$ is a vector bundle over Z. Show that if Z is closed in M, then every section of $E|_Z$ extends to a section of E.

Problems 3abc of Problem Set 1 can all be viewed as special cases of this problem (although we have not yet shown that $\bigwedge^k T^*M$ is a vector bundle over M). That is why the arguments for all three parts of that previous problem, as well as for the problem above, are essentially identical.

7. Let M be a manifold. For $k \geq 1$, let $M_k(\mathbf{R})$ denote the space of $k \times k$ matrices with real entries.

(a) Let $U \subset M$ be open, let $k \geq 1$, and let $A : U \to M_k(\mathbf{R})$ be a function. Define $F : U \times \mathbf{R}^k \to U \times \mathbf{R}^k$ by $F(p, v) = (p,\ A(p)v)$. Show that F is smooth if and only if the map A is smooth.

(b) Let E, E' be vector bundles over M and let $F : E \to E'$ be a bundle isomorphism (a smooth map that, for each $p \in M$, restricts to an isomorphism $E_p \to E'_p$). Prove that F is a diffeomorphism.

In particular, the total spaces E, E' of isomorphic vector bundles over M are diffeomorphic manifolds. This is purely of *global* interest and importance, since for *any* two rank-k vector bundles E, E' over M, every $p \in M$ has an open neighborhood U such that $\pi^{-1}(U)$ and $(\pi')^{-1}(U)$ are diffeomorphic to $U \times \mathbf{R}^k$ and are isomorphic as vector bundles over U. (Here π, π' are the projection maps of E, E'.)

Remarks. (1) A simplified version of the argument for part (b) shows that if F is merely a *continuous* map $F : E \to E'$ that restricts to an isomorphism $E_p \to E'_p$ for each $p \in M$, then F is a *homeo*morphism. (2) It can be shown that if E, E' are (smooth) vector bundles over M and and there is a continuous map $F : E \to E'$ that restricts to an isomorphism $E_p \to E'_p$ for each $p \in M$ (hence is a homeomorphism, by the first remark), then there is also a smooth such map (hence a diffeomorphism, by part (b) of the problem). This is markedly different from the analogous statement for manifolds. There are smooth manifolds that are homeomorphic but not diffeomorphic.

8. **Sub-bundles and quotient bundles**. Let E be a rank-k vector bundle over a manifold M. A (vector) sub-bundle of E is a subset of E that inherits from E the structure of a vector bundle over M. More precisely, given a subset $E' \subset E$, for each $p \in M$ define $E'_p = E' \cap E_p$, and call a vector-bundle chart (V, ψ) of E *adapted to* E' if for all $p \in V$ and some $r \in \{0, \ldots, k\}$ the map $\psi_p|_{E'_p}$ carries E'_p bijectively to $\mathbf{R}^r \times \{0 \in \mathbf{R}^{k-r}\}$. (If $r = 0$, omit the first factor in this Cartesian product; if $r = k$ omit the second factor. The cases $r = 0$ and $r = k$ are not very interesting, of course.) Note that such a chart can exist only if (for all $p \in V$) F_p is a r-dimensional vector subspace of E_p, in which $\psi_p|_{E'_p}$ carries E'_p *isomorphically* to $\mathbf{R}^r \times \{0 \in \mathbf{R}^{k-r}\}$. Call a vector-bundle atlas of E *adapted to* E' if every chart is adapted to E'. A *sub-bundle* of E is a subset E' for which E has an adapted vector-bundle atlas. Note that the integer r above is automatically constant on connected components of M. If it is constant throughout M (in case M is not connected), which is the only case we will consider, then by post-composing the maps ψ above with id. $\times$ {natural projection $\mathbf{R}^k \to \mathbf{R}^r \times \{0\}$, we obtain a vector-bundle atlas of

rank r for E'.

(a) Let E' be a rank-r subbundle of E, and let $(V_\alpha, \psi_\alpha), (V_\beta, \psi_\beta)$ be adapted vector-bundle charts with $V_\alpha \bigcap V_\beta \neq \emptyset$. Assume that $0 < r < k$. The transition function $h_{\alpha\beta}$ for E can be written in block form as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is an $r \times r$ matrix, B is $r \times (k-r)$, C is $(k-r) \times r$, and D is $(k-r) \times (k-r)$. What are the most general things you can say about the matrices A, B, C, and D?

(b) Let E' be a rank-r sub-bundle of E. For each $p \in M$, the quotient space E_p/E'_p is a vector space of dimension r. (i) Show that the set $\coprod_{p \in M}(E_p/E'_p)$ carries a natural vector-bundle structure of rank $k-r$. (One way to do this is to construct a vector-bundle atlas of rank $k-r$ out of an adapted atlas for the sub-bundle E'. For this, you should find part (a) helpful.) This bundle, denoted E/E', is called the *quotient bundle* for the given bundle E and sub-bundle E'. (ii) Show that there is a bundle homomorphism $E \to E/E'$ that restricts to the natural quotient map $E_p \to E_p/E'_p$ for each $p \in M$.

(c) Let $Z \subset M$ be a submanifold. Then $TM|_Z$ is a vector bundle over Z. (i) Show that TZ is a sub-bundle of $TM|_Z$. (ii) If M is equipped with a Riemannian metric g, then, as a point-set, the *geometric normal bundle* of Z is defined as in class. Let us now denote this as $\nu_{\mathrm{geom}}(Z)$. Show that $\nu_{\mathrm{geom}}(Z)$ is a sub-bundle of $TM|_Z$. Note that if we change the Riemannian metric, the underlying set $\nu_{\mathrm{geom}}(Z) \subset TM|_Z$ can change. (iii) The *algebraic normal bundle of Z* is the quotient bundle $\nu_{\mathrm{alg}}(Z) = (TM|_Z)/TZ$. This is more canonical than the geometric normal bundle, in the sense that no Riemannian metric is needed to define it. But show that for every Riemannian metric, the geometric normal bundle of Z is isomorphic to the algebraic normal bundle of Z.