Differential Geometry—MTG 6257—Spring 2018
Problem Set 3
Due-date: Wednesday, 4/4/17

Required problems (to be handed in): 1a, 2b(i), 3b, 6abe.

In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Optional problems: All the ones that are not required.

Required reading: Problem 2e, the setup of problem 5, all parts of problem 6, problem 7, and all the Remarks in this assignment.

1. **Induced connections on dual bundles.** Let ∇ be a connection on a vector bundle E over a manifold M.

(a) Show that there is a unique connection ∇' on the dual bundle E^* such that

$$X(\langle \xi, s\rangle) = \langle {\nabla'}_X \xi, s\rangle + \langle \xi, \nabla_X s\rangle \tag{1.1}$$

for all vector fields X and all sections ξ, s of E, E^* respectively. (In (1.1), the dual-pairings are taken pointwise, of course.)

(b) Let $\{s_\alpha\}_{\alpha=1}^k$ be a local basis of sections of E and let $\{\xi^\alpha\}$ be the local basis of sections of E^* dual to $\{s_\alpha\}$. Let Θ, Θ' be the connection forms of ∇, ∇' with respect to these local bases. Since we are using upper indices for the basis sections of E^*, we write the first index of Θ' downstairs and the second index upstairs[1]:

$$\nabla' \xi^\beta = \xi^\alpha \otimes {(\Theta')_\alpha}^\beta$$

Show that Θ' is the negative transpose of Θ, in the sense that ${(\Theta')_\alpha}^\beta = -{\Theta^\beta}_\alpha$.

(c) Show that for all $p \in M$ and $X, Y \in T_pM$, the endomorphism $F^{\nabla'}(X,Y) : E_p^* \to E_p^*$ is the negative of the natural adjoint of $F^\nabla(X,Y) : E_p \to E_p$.

2. **Induced connections on direct sums, tensor products, and homomorphism bundles.** Let $\nabla^{(1)}, \nabla^{(2)}$ be connections on vector bundles E_1, E_2 over a manifold M. Let $U \subset M$ be an open set over which both E_1 and E_2 are trivial, let $\{s_\alpha\}_{\alpha=1}^{k_1}, \{t_\mu\}_{\mu=1}^{k_2}$ be bases of sections of E_1, E_2 (respectively) over U, and let $\Theta^{(1)}, \Theta^{(2)}$ be the corresponding connection forms.

(a) The *direct sum connection* ∇ on $E_1 \oplus E_2$ is defined by $\nabla \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \nabla^{(1)} s \\ \nabla^{(2)} t \end{pmatrix}$, i.e. $\nabla_X \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \nabla^{(1)}_X s \\ \nabla^{(2)}_X t \end{pmatrix}$, where $s \in \Gamma(E_1), t \in \Gamma(E_2)$, and $X \in \Gamma(TM)$. (It is

[1] By default, LaTeX stacks superscripts directly on top of subscripts, as in B^k_j, making it impossible to distinguish which is the first index and which is the second. One way to produce, say, ${B^i}_j$, is \$ { B^i}_ j \$.

easy to check that this *does* define a connection.) Find the connection form of ∇ with respect to the basis $\begin{pmatrix} s_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} s_{k_1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ t_{k_2} \end{pmatrix}$.

(b) (i) Show that there is a unique connection ∇ on $E_1 \otimes E_2$ such that

$$\nabla_X(s \otimes t) = (\nabla_X^{(1)} s) \otimes t + s \otimes \nabla_X^{(2)} t \tag{1.2}$$

for all $s \in \Gamma(E_1), t \in \Gamma(E_2)$, and $X \in \Gamma(TM)$. Note that this cannot be deduced from applying the universal property of tensor products to $\Gamma(E_1) \otimes \Gamma(E_2)$, since "$s \otimes t$" denotes the *pointwise* tensor product $p \mapsto s(p) \otimes t(p)$. If we attempt to take (1.2) as a *definition* of ∇, it is not obvious without a little computation that $\nabla_X(s \otimes t)$ is well-defined, not just because elements of the form $s_p \otimes t_p$ don't form a basis of the vector space $E_{1,p} \otimes E_{2,p}$, but because $s \otimes t = fs \otimes (1/f)t$ for any nonvanishing $f \in C^\infty(M)$.

We call ∇ the *tensor product connection* determined by $\nabla^{(1)}$ and $\nabla^{(2)}$.

(ii) Find the connection form of ∇ with respect to the local basis of sections $\{s_\alpha \otimes t_\mu\}$ of $E_1 \otimes E_2$.

(iii) Show that the curvature F^∇ satisfies

$$F^\nabla(X,Y)(s \otimes t) = (F^{\nabla^{(1)}}(X,Y)s) \otimes t + s \otimes (F^{\nabla^{(2)}}(X,Y)t) \tag{1.3}$$

for all $s \in \Gamma(E_1), t \in \Gamma(E_2)$, and $X, Y \in \Gamma(TM)$. We may write (1.3) symbolically as

$$F^\nabla = F^{\nabla^{(1)}} \otimes \mathrm{id}_{E_2} + \mathrm{id}_{\mathrm{E}_1} \otimes F^{\nabla^{(2)}}.$$

(c) Combining problems 2b and 1, there is an induced connection ∇ on $E_2 \otimes E_1^* \underset{\text{can.}}{\cong} \mathrm{Hom}(E_1, E_2)$. Show that this connection satisfies

$$(\nabla_X A)(s) = \nabla_X^{(2)}(A(s)) - A(\nabla_X^{(1)}(s))$$

for all $A \in \Gamma(\mathrm{Hom}(E_1, E_2))$, $s \in \Gamma(E_1)$, and $X \in \Gamma(TM)$. Note that this can be written as the Leibnizian-looking formula

$$\nabla_X^{(2)}(A(s)) = (\nabla_X A)(s) + A(\nabla_X^{(1)}(s)).$$

(d) Show that the induced connection ∇ on $\mathrm{End}(E_1) \underset{\text{can.}}{\cong} E_1 \otimes E_1^*$ satisfies

$$F^\nabla(X,Y)A = [F^{\nabla^{(1)}}(X,Y), A] := F^{\nabla^{(1)}}(X,Y) \circ A - A \circ F^{\nabla^{(1)}}(X,Y)$$

for all $A \in \Gamma(\mathrm{End}(E_1))$ and $X, Y \in \Gamma(TM)$.

(e) Let $\mathbf{R}_M$ denote the product bundle $M \times \mathbf{R} \to M$, and let $\nabla^{(0)}$ denote the canonical connection. (Recall from class that $\nabla^{(0)} f = df$.) The bundles $\mathbf{R}_M \otimes E_1$ and $E_1 \otimes \mathbf{R}_M$ are canonically isomorphic to E_1. Show that these canonical isomorphisms carry the tensor-product connections on $\mathbf{R}_M \otimes E_1$ and $E_1 \otimes \mathbf{R}_M$, induced by $\nabla^{(0)}$ and $\nabla^{(1)}$, to the connection $\nabla^{(1)}$.

Remark 1.1 The construction of direct-sum connections and tensor-product connections extends in an obvious way to direct sums and tensor products of more than two vector bundles. In particular, a connection on a vector bundle E induces a connection on any bundle of the form $E_1 \otimes \ldots \otimes E_k$ ($k \geq 1$) where for each i, the bundle E_i is either E or E^*. It is too cumbersome to have distinct notation for each of these induced connections (as we did in problems 1 and 2). Hence, if ∇ is a connection on E, we generally use the same notation ∇ for the induced connection on any of these bundles. In any term in a formula or equation, context—the type of section being differentiated—makes clear which connection is being used. ■

Remark 1.2 (A convention used below) A *tensor bundle* over a manifold M is any bundle of the form $E_1 \otimes \ldots \otimes E_k$ ($k \geq 1$), where for each i, the bundle E_i is either TM or T^*M; if $k = 1$ we also allow the trivial product bundle $\mathbf{R}_M = M \times \mathbf{R} \to M$. (Because of the canonical isomorphisms mentioned in problem 2(e), we gain no new bundles by allowing $E_i = \mathbf{R}_M$ if $k > 1$, but there is no harm in allowing it.) A connection ∇ on TM then induces a connection (also denoted ∇) on every tensor bundle over M, provided we define which connection to use on the trivial bundle $\mathbf{R}_M$. In view of problem 2(e), in the context of induced connections on tensor bundles, we define the "induced" connection ∇ on $\mathbf{R}_M$ to be the canonical connection on this product bundle (no matter what connection is used on TM).

With this convention, given a connection ∇ on TM, the collection of induced connections on tensor bundles is "Leibnizian with respect to contractions" in the sense that (1.1) holds with a single symbol "∇", and "Leibnizian with respect to tensor products" in the sense that (1.2) holds with a single symbol "∇". ■

Remark 1.3 We also sometimes refer to sub-bundles and direct sums of tensor bundles as tensor bundles, but do not make that generalization in this homework assignment. ■

3. Let ∇ be a connection on a vector bundle E over a manifold M. By problems 1 and 2, ∇ induces a connection on $E^* \otimes E^*$.

(a) Show that the induced connection ∇ on $E^* \otimes E^*$ preserves the sub-bundles $\mathrm{Sym}^2(E^*)$ and $\bigwedge^2(E^*)$ in the following sense: if s is a section of either of these sub-bundles, and X is a vector field on M, then $\nabla_X s$ is a section of the same sub-bundle. Thus, the restriction of ∇ to sections of either of these sub-bundles is a connection on that sub-bundle.

(b) Let g be a Riemannian metric (in the vector-bundle sense) on E. Show that the connection ∇ on E respects g if and only if $\nabla g = 0$. Thus if ∇ respects g, then g is covariantly constant with respect to the induced connection on $E^* \otimes E^*$ (whence the terminology "∇ preserves g").

(c) In the setting of part (b), let $\mathsf{g} \in \Gamma(\mathrm{Hom}(E, E^*))$ be the bundle homomorphism determined by the metric g. Show that if g is covariantly constant, then so is g:

$$\nabla_X(\mathsf{g}(s)) = \mathsf{g}(\nabla_X s) \quad \text{for all } s \in \Gamma(E), X \in \Gamma(TM).$$

4. **The covariant Hessian.** Let ∇^E be a connection on a vector bundle E over M, and let ∇^M be a connection on TM. For all vector fields X, Y and all $s \in \Gamma(E)$, let

$$(\widetilde{H}s)(X,Y) = \nabla^E_X \nabla^E_Y s - \nabla^E_{\nabla^M_X Y} s.$$

(In case the last term of the formula is hard to read: in that term, "$\nabla^M_X Y$ is a subscript to ∇^E; at each $p \in M$ we are differentiating s in the direction $\nabla^M_X Y\big|_p$, using the connection ∇^E.

(a) Show that $(\widetilde{H}s)(X,Y)$ is $\mathcal{F}$-bilinear in (X,Y). Hence, for each s, the map $(X,Y) \mapsto (\widetilde{H}s)(X,Y)$ is tensorial, and therefore defines a section Hs of $E \otimes T^*M \otimes T^*M$.

The section Hs is called the *covariant Hessian of* s with respect to the connections ∇^E and ∇^M. If (M,g) is Riemannian, and you see the term "covariant Hessian" used without the connections ∇^E and ∇^M having both been specified explicitly, the writer is probably using the following conventions:

- ∇^M is the Levi-Civita connection on (M,g).
- If E is a tensor bundle over M, then ∇^E is the one induced by the Levi-Civita connection. Note that for the product bundle $\mathbf{R}_M$, this means that the canonical connection ($\nabla f = df$) is used, so the covariant Hessian of $f \in C^\infty(M)$ is given by $Hf(X,Y) = X(Y(f)) - (\nabla_X Y)(f)$, where ∇ is the Levi-Civita connection.

(b) Show that if ∇^M is torsion-free, then for all sections s and vector fields X, Y,

$$Hs(X,Y) - Hs(Y,X) = F^{\nabla^E}(X,Y)s.$$

Thus, the left-hand side is tensorial in s, even though neither term individually is tensorial in s.

(c) Following the conventions mentioned above, show that on a Riemannian manifold, the covariant Hessian of any function $f \in C^\infty(M)$ is a *symmetric* tensor field. (All that is needed for this symmetry is that we use a torsion-free connection on TM; the metric does not enter the argument.)

5. **Higher-order covariant derivatives.** Let ∇ be a connection on TM. Using the convention in Remark 1.2, we have an induced connection denoted ∇ on every

tensor bundle over M. Since a connection on any vector bundle E maps $\Gamma(E)$ to $\Gamma(E \otimes T^*M)$, we therefore have an infinite sequence of maps

$$C^\infty(M) = \Gamma(\mathbf{R}_M) \xrightarrow{\nabla = d} \Gamma(T^*M) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M \otimes T^*M) \xrightarrow{\nabla} \dots . \tag{1.4}$$

(For any tensor bundle E we have a similar sequence, with $\Gamma(\mathbf{R}_M)$ replaced by $\Gamma(E)$, and with $(T^*M)^{\otimes k}$ replaced by $E \otimes (T^*M)^{\otimes k}$.) In particular, for $f \in C^\infty(M)$, a connection on TM allows us to define $\nabla\nabla f, \nabla\nabla\nabla f$, etc.

(a) Show that if the connection ∇ is a torsion-free, then $\nabla\nabla f$ is the covariant Hessian defined in problem 3.

If you do part (a) carefully, you will likely find that you are actually doing part (b), and then deducing part (a) using problem 3c. I've stated part (a) separately since it's more forgiving of an easily-made mistake.

(b) More generally, show that if the connection ∇ is arbitrary, then $\nabla\nabla f$ is still the covariant Hessian up to a "transpose":

$$(\nabla\nabla f)(X, Y) = Hf(Y, X). \tag{1.5}$$

(This is true with f replaced by a section of any tensor bundle; I'm just giving you the simplest case for homework. An analog is also true for sections of an *arbitrary* vector bundle E, except that we need to specify two initial connections, ∇^E and ∇^M, to define what "∇" is going to mean beyond the first map in the sequence analogous to (1.4).)

6. **Curvature of submanifolds.** Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold, let M be a submanifold, let $j : M \to \tilde{M}$ be the inclusion map, and let $g = j^*\tilde{g}$ (the induced metric on M). Let $\tilde{\nabla}, \nabla$ be the Levi-Civita connections on $(\tilde{M}, \tilde{g})$ and (M, g) respectively. Recall that for every vector field X on M and any $p \in M$, there is an M-open neighborhood U of p such that $X|_U$ extends to a vector field on some $\tilde{M}$-open neighborhood of p. (Here and below, "vector field on M" means a *tangent* vector field—a section of TM, not just a section of $T\tilde{M}|_M$.)

At each $p \in M$, let $\pi_{\tan}$, $\pi_{\rm nor}$ denote orthogonal projection from $T_p\tilde{M}$ to the tangent space T_pM and normal space ν_pM respectively.

(a) Let X, Y be vector fields on M, and let $U \subset M$ be an M-open neighborhood small enough that $X|_U, Y|_U$ extend to vector fields $\tilde{X}, \tilde{Y}$ on some $\tilde{M}$-open set. (We will not need $\tilde{X}$ till part (b).) Show that at each point of U we have

$$\nabla_X Y = \pi_{\tan}(\tilde{\nabla}_X \tilde{Y}),$$

independent of the choice of local extension $\tilde{Y}$.

(b) Notation as in part (a). Show that at each $p \in U$ we have

$$\pi_{\rm nor}\left(\tilde{\nabla}_X \tilde{Y}\right) = \pi_{\rm nor}\left(\tilde{\nabla}_Y \tilde{X}\right). \tag{1.6}$$

The left-hand side of (1.6) is clearly $\mathcal{F}(M)$-linear in X and independent of the choice of local extension $\tilde{X}$, so the right-hand side must have the same property; similarly both sides are $\mathcal{F}(M)$-linear in Y and independent of the choice of $\tilde{Y}$. Hence each side is $\mathcal{F}(M)$-bilinear and therefore tensorial, defining at each point $p \in M$ a symmetric, $\nu_p M$-valued bilinear form h_p on T_pM. As p varies over M we obtain a section $h \in \Gamma(\nu M \otimes \mathrm{Sym}^2(T^*M))$ such that

$$h(X, Y) = \pi_{\mathrm{nor}}(\widetilde{\nabla}_X \tilde{Y}). \tag{1.7}$$

for all vector fields X, Y on M and local extensions $\tilde{Y}$ of Y. The object h is called the *second fundamental form* of the submanifold M. (The *first fundamental form* is simply the induced metric g.)

Remark 1.4 (1) Sometimes we call the object h defined in (1.7) the "vector-valued second fundamental form" to distinguish it from the related scalar-valued object in part (d) below. (2) The second fundamental form of a submanifold is sometimes called the *extrinsic curvature*, for reasons mentioned in part (d). ■

(c) Show that for all vector fields X, Y, Z, W on M, at each point of M we have the *Gauss equation*

$$\begin{aligned} g(R^M(X,Y)Z, W) &= \tilde{g}(R^{\tilde{M}}(X,Y)Z, W) \\ &\quad + \tilde{g}(h(X,W), h(Y,Z)) - \tilde{g}(h(X,Z), h(Y,W)), \end{aligned} \tag{1.8}$$

where $R^M(\cdot,\cdot)\cdot$ and $R^{\tilde{M}}(\cdot,\cdot)\cdot$ denote the Riemann tensors of (M, g) and $(\tilde{M}, \tilde{g})$ respectively.

Remark 1.5 Observe that if the ambient manifold $(\tilde{M}, \tilde{g})$ is Euclidean space, then the first term on the right-hand side of (1.8) is zero. Hence, in this case, the Riemann tensor of the abstract Riemannian manifold (M, g)—an intrinsic curvature defined purely from the metric g, without ever embedding M into a larger manifold—is completely determined by the extrinsic curvature h.

(d) Suppose now that the codimension of M is 1. Then, locally, there are exactly two unit normal vector fields, say $\pm N$. Then for all $p \in M$ and $X, Y \in T_pM$ we have

$$h(X, Y) \;=\; \tilde{g}(h(X,Y), N)\, N =: \hat{h}(X,Y)\, N.$$

We call $\hat{h}(X, Y)$ the (locally defined, scalar-valued) *second fundamental form determined by* N. (There are two such locally-defined forms, each the negative of the other.) Show that

$$\hat{h}(X, Y) = -\tilde{g}(\widetilde{\nabla}_X N, Y). \tag{1.9}$$

Thus, the second fundamental form $\hat{h}$ measures the "bending" of the unit normal N as we move along M, as viewed from within the ambient manifold $\tilde{M}$. This bending is *extrinsic to* (M, g): to a creature whose universe is the abstract Riemannian manifold (M, g), there *is* no larger ambient space, no unit normal, and no second fundamental form.

Remark 1.6 One of Gauss's brilliant discoveries was that there is such a thing as *intrinsic* curvature. Abstract manifolds had yet to be discovered, and the curvature of a surface in $\mathbf{R}^3$ was thought of only as the second fundamental form (or the related "shape operators" mentioned in Remark 1.9). In his *Theorema Egregium*, Gauss proved that what is now called Gaussian curvature is an isometric invariant: if $F : (M, g) \to (M', g')$ is what we now call an isometry (of surfaces), then F carries the Gaussian curvature of (M, g) to the Gaussian curvature of (M', g'). It is possible for surfaces M and M' in $\mathbf{R}^3$ to be isometric without having the same *extrinsic* geometry, i.e. without one being the image of the other under a rigid motion of $\mathbf{R}^3$. The invariance of Gaussian curvature under isometries means that it is something *intrinsic* to a Riemannian 2-manifold (M, g), definable without any embedding in $\mathbf{R}^3$. ■

Remark 1.7 The number $\hat{h}(X, Y)|_p$ depends on which of the two unit normals we call N_p, but the vector $h(X, Y)|_p$ does not. If M is connected and the normal bundle is orientable, then there are exactly two globally-defined unit normal vector fields N, hence two globally defined scalar-valued second fundamental forms (differing by a sign).

The modifier "scalar-valued" is often omitted to shorten wording when context makes the meaning clear. It is always omitted in elementary courses on surfaces in $\mathbf{R}^3$. ■

Remark 1.8 If $(\tilde{M}, \tilde{g})$ is Euclidean space $\mathbf{R}^n$, then $\hat{h} \equiv 0$ on a connected open set $U \subset M$ if and only if N is represented by a constant $\mathbf{R}^n$-valued function on U; equivalently, if and only if U is contained in a hyperplane. Geometrically, it is natural to interpret the non-constancy of N as some sort of curving of the submanifold M; hence the term "extrinsic *curvature*". ■

Remark 1.9 Observe that in the codimension-1 case, the Gauss equation (1.8) simplifies to

$$g(R^M(X, Y)Z, W) = \tilde{g}(R^{\tilde{M}}(X, Y)Z, W) + \hat{h}(X, W)\,\hat{h}(Y, Z) - \hat{h}(X, Z)\,\hat{h}(Y, W), \tag{1.10}$$

regardless of which unit normal is used to define $\hat{h}$. If $(\tilde{M}, \tilde{g})$ is Euclidean space, then the ambient curvature is zero, and (1.10) simplifies even more:

$$g(R^M(X, Y)Z, W) = \hat{h}(X, W)\,\hat{h}(Y, Z) - \hat{h}(X, Z)\,\hat{h}(Y, W). \tag{1.11}$$

Finally, if M is a surface in Euclidean 3-space, then (1.11) gives us a simple formula for Gaussian curvature: for $p \in M$ and any orthnormal basis $\{e_1, e_2\}$ of T_pM,

$$K(p) = g(R^M(e_1, e_2)e_2, e_1) = \begin{vmatrix} \hat{h}(e_1, e_1) & \hat{h}(e_1, e_2) \\ \hat{h}(e_2, e_1) & \hat{h}(e_2, e_2) \end{vmatrix} = \det(S_p), \tag{1.12}$$

where the *shape operator* $S_p : T_pM \to T_pM$ (determined by the choice of N) is the self-adjoint linear map defined by the equation $\hat{h}(X,Y) = g(S_p(X), Y)$. By considering the eigenvalues of this shape operator, one can show that M is "bowl-shaped" at p if $K(p) > 0$ and "saddle-shaped" at p if $K(p) < 0$. (The case $K(p) = 0$ is indeterminate, analogously to the Calculus-3 second-derivative test for local extrema of a real-valued function of two variables.) ■

(e) Consider the example $M = S^n \subset \mathbf{R}^{n+1} = \tilde{M}$, where $\mathbf{R}^{n+1}$ is given its standard Riemannian metric. Let N be the outward-pointing unit normal vector field on M. Show that for every vector field X on M we have $\widetilde{\nabla}_X N = X$, and deduce from this the following facts (simply writing R instead of R^M)

(i) $g(R(X,Y)Z, W) = g(X,W)g(Y,Z) - g(X,Z)g(Y,W)$ for all $p \in S^n$ and all $X, Y, Z, W \in T_pS^n$.

(ii) $R(X,Y)Z = g(Y,Z)X - g(X,Z)Y$ for all $p \in S^n$ and all $X, Y, Z \in T_pS^n$.

(iii) The sectional-curvature function $\sigma : G_2(TS^n) \to \mathbf{R}$ is the constant function 1.

7. **Hyperbolic space.** Let M be the open ball of radius 1 in $\mathbf{R}^n$, centered at the origin. Let $r : \mathbf{R}^n \to \mathbf{R}$ denote Euclidean distance to the origin. Let g_{Euc} be the standard Riemannian metric on $\mathbf{R}^n$, restricted to the open set M. Then define a metric g on M by

$$g = \frac{4}{(1-r^2)^2} g_{\text{Euc}} \ . \tag{1.13}$$

Show that M has constant curvature -1 (see Remark below). This Riemannian manifold is called the *Poincaré disk* or (the Poincaré model of) *hyperbolic n-space.*

Remark 1.10 A Riemannian manifold (M, g) is said to have *constant curvature*, or *constant sectional curvature*, if the sectional-curvature function $G_2(TM) \to \mathbf{R}$ is constant. These are *very* exceptional manifolds. For every dimension $n \geq 2$, and for every real number K, up to isometry there is exactly one simply connected, complete Riemannian manifold of constant curvature K. (We have not defined what "complete" means for a Riemannian manifold. That's coming after we define the metric-space structure.) For $K > 0$ this constant-curvature manifold is the sphere of radius $1/\sqrt{K}$ in Euclidean space; for $K = 0$ it is Euclidean space; and for $K < 0$ it is a slightly modified version of the Poincaré disk (just multiply the metric in (1.13) by $1/\sqrt{|K|}$).

If you redo your computations for problem 7 with "$1 - r^2$" in (1.13) replaced by "$1 + r^2$" (and remove what then becomes the unnecessary restriction $r < 1$), you'll find that the resulting metric has constant curvature 1. This metric on $\mathbf{R}^n$ is exactly the pullback of the standard metric on S^n by stereographic projection. ■