Differential Geometry—MTG 6257—Spring 2018
Problem Set 4
Due-date: Wednesday, 4/25/18

Required problems (to be handed in): 2bc, 3, 5c, 5d(i).

In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Required reading: Remark 1.1; problem 2de; Remarks 1.3 and 1.4; problem 5ab; Remark 1.5; the result of problem 6d; Remarks 1.6 and 1.7; problem 7a; problem 8.

Optional problems: All the ones that are not required.

1. Lemma for use in later problem(s). Let $\{y^i\}$ be standard coordinates on $\mathbf{R}^n$, let $\omega \in \Omega^{n-1}(S^{n-1})$ be the standard volume form, and let $\mathrm{Vol}(S^{n-1}) = \int_{S^{n-1}} \omega$ (the volume of the standard, Euclidean, unit sphere). Show that for all $i, j \in \{1, \dots, n\}$,

$$\int_{S^{n-1}} y^i y^j \ \omega = \frac{1}{n}\, \delta_{ij}\, \mathrm{Vol}(S^{n-1}).$$

(This can be done without any trigonometric integrals.)

2. Ricci tensor and scalar curvature. Let (M, g) be a Riemannian manifold. For each $p \in M$ and $X, Y \in T_pM$, the Riemann tensor defines a linear map $T_pM \to T_pM$ by $Z \mapsto R(X, Z)Y$. Define

$$\mathsf{Ric}(X, Y) = \mathsf{Ric}|_p(X, Y) = \mathrm{tr}(Z \mapsto R(Z, X)Y),$$

where "tr" denotes the trace. (This trace-operation is also called *contraction* of the first and third factors of the tensor bundle $TM \otimes T^*M \otimes T^*M \otimes T^*M$.) Thus, if $\{e_i\}$ is an arbitrary basis of T_pM and $\{\theta^i\}$ is the dual basis of T_p^*M,

$$\mathsf{Ric}(X, Y) = \langle \theta^i, R(e_i, X)Y \rangle.$$

Clearly the map $(X, Y) \mapsto \mathsf{Ric}|_p(X, Y)$ is bilinear, so $\mathsf{Ric}|_p$ is an element of $T_p^*M \otimes T_p^*M$. This tensor is called the *Ricci tensor* at p. Letting p vary, it is easily seen that $\mathsf{Ric}|_p$ depends smoothly on p, so Ric becomes a tensor field on M, called the *Ricci tensor* (field) or the *Ricci curvature.*

(a) Show that with $p, \{e_i\}, \{\theta^i\}$ as above, the Ricci tensor at p is given by

$$\begin{aligned} \mathsf{Ric} &= R_{jl}\, \theta^j \otimes \theta^l, \\ &\quad \text{where } R_{jl} = R^i{}_{jil} \end{aligned}$$

and where $\{R^i{}_{jkl}\}$ are the components of the Riemann tensor at p with respect to the given bases.

(b) Show that the Ricci tensor is a *symmetric* tensor field: for all $p \in M$ and all $X, Y \in T_pM$, we have $\mathsf{Ric}(X,Y) = \mathsf{Ric}(Y,X)$.

Suggestion: Compute the trace defining $\mathsf{Ric}(X,Y)$ using an orthonormal basis of T_pM. Contraction with θ^i then becomes inner product with e_i.

(c) Below, for any normed vector space V, we write $S(V)$ for the unit sphere centered at the origin.

Assume that $n = \dim(M) \geq 2$. Recall that, at each p, the sectional curvature of M at p is a map $G_2(T_pM) \to \mathbf{R}$, $\mathcal{P} \mapsto \sigma(\mathcal{P})$. For $X \in S(T_pM)$ let $X^\perp = \{Y \in T_pM : Y \perp X\}$. Let $G_2^X(T_pM) \subset G_2(T_pM)$ denote the set of all 2-planes in T_pM that contain X. There is a two-to-one map

$$\begin{aligned} \pi_X : S(X^\perp) &\to G_2^X(T_pM), \\ \pi_X(Y) &= \mathcal{P}(X,Y) := \mathrm{span}\{X,Y\}. \end{aligned}$$

The vector space $X^\perp$ is a Riemannian manifold with the standard Riemannian metric determined by $g_p|_{X^\perp}$; thus $S(X^\perp)$ inherits a Riemannian metric. Orienting $X^\perp$ arbitrarily, and giving S^{n-1} the induced orientation, we then obtain a volume form form ω_{n-2} on $S(X^\perp)$. (The subscript here is just a reminder of the dimension of $S(X^\perp)$.) Show that for $X \in S(T_pM)$,

$$\int_{S(X^\perp)} (\sigma \circ \pi_X)\, \omega_{n-2} = \int_{S(X^\perp)} \sigma(\mathcal{P}(X,\cdot))\, \omega_{n-2} = \frac{\mathrm{Vol}(S^{n-2})}{n-1} \mathsf{Ric}(X,X). \tag{1.1}$$

Remark 1.1 Hence

$$\frac{1}{n-1}\, \mathsf{Ric}(X,X) = \frac{1}{\mathrm{Vol}(S(X^\perp))} \int_{S(X^\perp)} (\sigma \circ \pi_X)\, \omega_{n-2}\ . \tag{1.2}$$

Thus, up to the normalization constant $\frac{1}{n-1}$, the quantity $\mathsf{Ric}(X,X)$ represents the *average sectional curvature among all two-planes in T_pM that contain X*.[1] ■

[1]The reason we integrated over $S(X^\perp)$ in (1.1) and (1.2), rather than over $G_2^X(T_pM)$, is that $G_2^X(T_pM)$ is diffeomorphic to the projective space $\mathbf{RP}^{n-2}$, which is not orientable when n is even. However, whether or not a Riemannian manifold (N, g_N) is orientable, the metric g_N induces a well-defined measure "$d\mu_N$" on N; it's simply something that we did not discuss in the non-orientable case (it's not a differential form in that case). Therefore for any finite-dimensional inner-product space W, the projectization $\mathbf{P}(W)$ has a Riemannian metric, hence Riemannian measure $d\mu$, induced the by the natural two-to-one covering map $\pi' : S(W) \to \mathbf{P}(W)$ and the standard Riemannian metric on $S(W)$. (Here we regard W as a Riemannian manifold with the standard Riemannian metric determined by the given inner product on W.) Using these facts it can be shown $\mathrm{Vol}(S(X^\perp)) = 2\mathrm{Vol}(G_2^X(T_pM))$ and that

$$\int_{S(X^\perp)} (\sigma \circ \pi_X)\, \omega = \int_{G_2^X(T_pM)} \sigma\ d\mu.$$

Thus (1.2) indeed represents the average value of the function $\sigma|_{G_2^X(T_pM)}$ with respect to the induced measure on $G_2^X(T_pM)$.

Remark 1.2 Recall that for any finite-dimensional vector space V, any symmetric bilinear form $h : V \times V \to \mathbf{R}$ is determined by its restriction to the diagonal: if we know $h(X,X)$ for *all* $X \in V$, then we know $h(X,Y)$ for all $X,Y \in V$. This follows from the *polarization identity*

$$h(X,Y) = \frac{h(X+Y,X+Y) - h(X-Y,X-Y)}{4} \ .$$

Furthermore, if V is equipped with a norm $\| \ \|$, then for all nonzero $X \in V$ we have $h(X,X) = \|X\|^2 h(\hat{X},\hat{X})$, where $\hat{X} = X/\|X\|$. Thus, in the presence of a norm, a symmetric bilinear form h can be completely recovered from the function f_h (notation just for this problem) that h determines on the unit sphere:

$$\begin{aligned} f_h : S(V) := \{X \in V : \|X\| = 1\} &\to \mathbf{R}, \\ X &\mapsto f_h(X) := h(X,X). \end{aligned}$$

In particular, for each $p \in M$, the function $f_{\mathsf{Ric}} : S(T_pM) \subset T_pM$ carries all the information of the Ricci tensor at p. ■

(d) Let $\mathsf{g}_p : T_pM \to T_p^*M$ be the isomorphism induced by the inner product g_p. For any tensor $h_p \in T_p^*M \otimes T_p^*M$, we define the *trace of* h_p *with respect to* g_p, denoted $\mathrm{tr}_{g_p}(h_p)$, to be the image of h_p under the following composition maps

$$T_p^*M \otimes T_p^*M \stackrel{\mathsf{g}_p{}^{-1}\otimes \mathrm{id.}}{\longrightarrow} \underset{\text{canon.}}{\cong} \mathrm{Hom}(T_pM, T_pM) \stackrel{\text{trace}}{\longrightarrow} \mathbf{R}.$$

Applying this pointwise to any $h \in \Gamma(\mathrm{Sym}^2(T^*M))$ gives a real-valued function $\mathrm{tr}_g(h) : M \to \mathbf{R}$.

Show that for h as above, $p \in M$, $\{e_i\}$ any basis of T_pM, $g_{\cdot\cdot}$ the matrix of g_p with respect to this basis, and $g^{\cdot\cdot} = (g_{\cdot\cdot})^{-1}$,

$$\mathrm{tr}_g(h)|_p = g^{ij}h_{ij} = h^i{}_i = h_i{}^i \ ,$$

where $h_{ij} = h(e_i, e_j)$.

(e) The *scalar curvature* or *Ricci scalar* is the real-valued function $\mathsf{R} = \mathrm{tr}_g(\mathsf{Ric})$ on M. Show that at each $p \in M$,

$$\mathsf{R}(p) = \frac{n}{\mathrm{Vol}(S^{n-1})} \int_{S(T_pM)} f_{\mathsf{Ric}} \ \omega_{n-1} \ ,$$

where f_{Ric} is as in Remark 1.2 and ω_{n-1} is the volume form on the sphere $S(T_pM)$ induced by the metric g_p and an arbitrary choice of orientation of T_pM.

Remark 1.3 Thus the "normalized scalar curvature" $\frac{1}{n}\mathsf{R}(p)$ is simply the average value of the function $S(T_pM) \to \mathbf{R}, X \mapsto \mathsf{Ric}(X, X)$. But for each $X \in S(T_pM)$, the quantity $\frac{1}{n-1} f_{\mathsf{Ric}}(X)$ is itself an average of sectional curvatures, so scalar curvature is sometimes thought of as a (normalized) "double average" of sectional curvatures. However, the word "double" can be eliminated: it can be shown that $\frac{1}{n(n-1)}\mathsf{R}(p)$ is the average value of the sectional-curvature function $\sigma_p : G_2(T_pM) \to \mathbf{R}$. ■

Remark 1.4 For $r > 0$ and $p \in M$, let $S_r^{n-1}(p), \bar{B}_r^n(p)$ denote, respectively, the sphere $\{q \in M \mid d_g(q, p) = r\}$ and the ball $\{q \in M \mid d_g(q, p) \leq r\}$, i.e. the sphere and closed ball of radius r and center p in the metric space (M, d_g). Curvature affects the growth-rate of the volumes of these spheres and balls with respect to the radius. A remarkable fact is that, if we compare the volumes of these spheres and balls to their Euclidean counterparts, then as $r \to 0$ (with p fixed), the leading-order corrections to the volumes as a function of r are governed entirely by the scalar curvature $\mathsf{R}(p)$. Specifically, letting S_r^{n-1} and $\bar{B}_r^n$ (with no "p") denote the Euclidean sphere and closed ball, as $r \to 0$ we have

$$\mathrm{Vol}(S_r^{n-1}(p)) = \mathrm{Vol}(S_r^{n-1}) \left(1 - \frac{1}{6n}\,\mathsf{R}(p) r^2 + O(r^3)\right) \tag{1.3}$$

and

$$\mathrm{Vol}(\bar{B}_r^n(p)) = \mathrm{Vol}(\bar{B}_r^n) \left(1 - \frac{1}{6(n+2)}\,\mathsf{R}(p) r^2 + O(r^3)\right) \tag{1.4}$$

(Needless to say, the Euclidean volume-dependencies on r are $\mathrm{Vol}(S_r^{n-1}) = \mathrm{Vol}(S_1^{n-1}) r^{n-1}$ and $\mathrm{Vol}(\bar{B}_r^n) = \mathrm{Vol}(\bar{B}_1^n) r^n$.) Equations (1.3) and (1.4) quantify, asymptotically as $r \to 0$, the statement that "larger (sectional) curvature means smaller balls and spheres." ■

3. Pullback of a metric-preserving connection. Let M, N be manifolds, (E, h) be a Riemannian vector bundle over M, and let $f : N \to M$ be smooth. Suppose that ∇ is a connection on E that preserves the metric h. (Here "metric" is used in the sense of vector bundles: h is a smooth field of inner products on the fibers of E, not the fibers of TM [unless $E = TM$]). Show that the pulled-back connection $f^*\nabla$ preserves the pulled-back metric $f^\sharp h$.

Remember that the definition of $f^\sharp h$ *does not involve any derivatives of* f; $(f^\sharp h)_p$ is simply the inner product on $E_{f(p)} \underset{\text{canon.}}{\cong} (f^*E)_p$.

4. The "wedge-bracket" operation. (a) Let V be a finite-dimensional vector space and let $\mathcal{A}$ be an algebra. (Our only applications will be the associative algebras

$\mathcal{A} = M_{k\times k}(\mathbf{R})$ and $\mathcal{A} = \mathrm{End}(W)$ for some finite-dimensional vector space W; you may assume $\mathcal{A}$ is one of these if it helps you understand this problem.) For $B, C \in \mathcal{A}$, we write $[B, C] = BC - CB$.

Show that for all $j, l \geq 0$, there is a unique bilinear map

$$[\cdot,\cdot] : (\mathcal{A} \otimes \textstyle\bigwedge^j V^*) \times (\mathcal{A} \otimes \bigwedge^l V^*) \to \mathcal{A} \otimes \bigwedge^{j+l} V^*$$

(the "wedge-bracket" operation) such that

$$[B \otimes \omega, C \otimes \eta] = BC \otimes (\omega \wedge \eta) - (-1)^{jl} CB \otimes (\eta \wedge \omega) = [B, C] \otimes (\omega \wedge \eta) \qquad (1.5)$$

for all $B, C \in \mathcal{A}, \omega \in \bigwedge^p V^*, \eta \in \bigwedge^l V^*$.

(b) Show that for $\xi \in \mathcal{A} \otimes \bigwedge^j V^*$ and $\zeta \in \mathcal{A} \otimes \bigwedge^l V^*$ we have

$$[\xi, \zeta] = (-1)^{jl+1}[\zeta, \xi]. \qquad (1.6)$$

(Thus the wedge-bracket operation is antisymmetric if either j or l is even, and symmetric if both j and l are odd.)

(c) Let E be a vector bundle over a manifold M and let $k \geq 1$. For $p \in M$, let $\mathcal{A}_p$ be either of the algebras $\mathrm{End}(E_p)$, $M_{k\times k}(\mathbf{R})$. Show that the wedge-bracket operation, applied pointwise, yields bilinear maps

$$[\cdot,\cdot] : \Omega^j(M; \mathrm{End}(E)) \times \Omega^l(M; \mathrm{End}(E)) \to \Omega^{j+l}(\mathrm{End}(E))$$

and

$$[\cdot,\cdot] : \Omega^j(M; M_{k\times k}(\mathbf{R})) \times \Omega^l(M; M_{k\times k}(\mathbf{R})) \to \Omega^{j+l}(M_{k\times k}(\mathbf{R})),$$

satisfying (1.5) and (1.6) pointwise.

5. Covariant exterior derivative. Let E be a vector bundle over a manifold M. As in class, we will use the abbreviated notation "$\Omega^j(E)$" for $\Omega^j(M; E) = \Gamma(E \otimes \bigwedge^j T^*M)$.

(a) Let $j, l \geq 0$.

(i) Show that there is a unique bilinear map $\wedge : \Omega^j(E) \times \Omega^l(M) \to \Omega^{j+l}(E)$, $(\alpha, \omega) \mapsto \alpha \wedge \omega$, satisfying

$$(s \otimes \eta)_p \wedge \omega_p = s_p \otimes (\eta \wedge \omega)_p \quad \text{for all } p \in M. \qquad (1.7)$$

(ii) Show that there is a unique bilinear map $\Omega^j(\mathrm{End}(E)) \times \Omega^l(E) \to \Omega^{j+l}(E)$ satisfying

$$((A \otimes \eta)_p, (s \otimes \omega)_p) \mapsto A_p(s_p) \otimes (\eta_p \wedge \omega_p) \quad \text{for all } p \in M. \qquad (1.8)$$

(In this equation, the endomorphism A_p is applied to the vector $s_p \in E_p$, while the $\bigwedge^* T_p^* M$-factors are wedged together.) Henceforth we omit the subscript p equations like (1.7) and (1.8), understanding that an equation like "$(s \otimes \eta) \wedge \omega = s \otimes (\eta \wedge \omega)$" is to be interpreted as a pointwise statement.

For $F \in \Omega^j(\mathrm{End}(E))$ and $\xi \in \Omega^l(E)$, we will write $F(\xi)$ for the image of (F, ξ) under the map defined pointwise by (1.8). Regrettably, the notation is not self-explanatory, but (unlike for the wedge-bracket operation in problem 4) I know of no wonderful notation for this combined endomorphism-evaluation/wedge-product operation.

For the rest of this problem, let ∇ be a connection on E.

(b) Show that there is a unique linear map $d_\nabla : \Omega^*(E) \to \Omega^*(E)$ that satisfies

$$d_\nabla(s \otimes \omega) = (\nabla s) \wedge \omega + s \otimes d\omega \tag{1.9}$$

for all $s \in \Gamma(E)$, $\omega \in \Omega^j(M)$, $j \geq 0$. We call d_∇ the *covariant exterior derivative* operator determined by ∇.

(c) Show that, for $j \geq 0$, the operator $d_\nabla : \Omega^j(E) \to \Omega^{j+1}(E)$ is not $\mathcal{F}$-linear, but that $d_\nabla \circ d_\nabla : \Omega^j(E) \to \Omega^{j+2}(E)$ *is* $\mathcal{F}$-linear.

(d) Let $F^\nabla \in \Omega^2(\mathrm{End}(E))$ be the curvature 2-form of ∇.

(i) Show that for every $s \in \Gamma(E)$, $d_\nabla d_\nabla s = F^\nabla(s)$, where the notation is as in (a)(ii) above (with $j = 0$).

(ii) Show, more generally, that for any $j \geq 0$ and $\xi \in \Omega^j(E)$, $d_\nabla d_\nabla \xi = F^\nabla(\xi)$.

Remark 1.5 Hence for a *flat* connection, the pair $(\Omega^*(E), d_\nabla)$ is a cochain complex, and cohomology is defined. Remember, however, that not every vector bundle admits a flat connection. For those that do, the cohomology groups (in a given degree) defined by different flat connections may not be isomorphic. ■

6. Bianchi identity. Let ∇ be a connection on a vector bundle E over a manifold M. As seen in the previous assignment, ∇ canonically induces a connection on $\mathrm{End}(E)$, which we will again denote ∇. Below, "matrix" always means $k \times k$ matrix, where $k = \mathrm{rank}(E)$, and $[\cdot, \cdot]$ denotes the wedge-bracket operation on $M_{k \times k}$-valued differential forms (see problem 4).

(a) Let $\{s_\alpha\}$ be a local basis of sections of E, say on the open set $U \subset M$, and let Θ be the corresponding connection form for the connection on E. For $A \in \Gamma(\mathrm{End}(E)|_U)$ let $\hat{A}$ be the matrix-valued function for which $\hat{A}(p)$ is the matrix of the endomorphism

A_p of E_p with respect to the basis $\{s_\alpha(p)\}$. Show that the corresponding matrix representation of $d_\nabla A$ is

$$(d_\nabla A)\hat{} = d\hat{A} + [\Theta, \hat{A}]. \tag{1.10}$$

(b) Let $\{s_\alpha\}, U, \Theta$ be as in (a), but now let $A \in \Omega^j(U; \mathrm{End}(E)|_U)$ for arbitrary j, and let $\hat{A} \in \Omega^1(U; M_{k\times k}(\mathbf{R}))$ be the corresponding matrix-valued j-form. Show that (1.10) still holds.

(c) Let $\{s_\alpha\}, U, \Theta$ be as in (i). Recall that the corresponding representation of the curvature $F^\nabla|_U$ as a matrix valued 2-form is $\hat{F} = d\Theta + \Theta\wedge\Theta$. Show that

$$d\hat{F} = -[\Theta, \hat{F}].$$

(d) Use parts (b) and (c) to deduce the *Bianchi identity*

$$d_\nabla F^\nabla = 0 \in \Omega^3(M; \mathrm{End}(E)) = \Gamma(\mathrm{End}(E) \otimes \textstyle\bigwedge^3 T^*M). \tag{1.11}$$

Remark 1.6 In the bundle

$$\mathrm{End}(E) \otimes \textstyle\bigwedge^3 T^*M = E \otimes E^* \otimes \bigwedge^3 T^*M \ \subset\ E \otimes E^* \otimes T^*M \otimes T^*M \otimes T^*M,$$

there is (for general E) no relation between the first two factors of the tensor product and the last three factors. However, if $E = TM$, then

$$\mathrm{End}(E) \otimes \textstyle\bigwedge^3 T^*M \ \subset\ TM \otimes T^*M \otimes T^*M \otimes T^*M \otimes T^*M,$$

a tensor bundle in which *all* the factors are related to each other. Given a section of this bundle, we can, for example, contract the first and third factors (as we did to get Ric from the Riemann tensor), obtaining a section of $T^*M \otimes T^*M \otimes T^*M$. We can then "raise an index" to obtain a section of $TM \otimes T^*M \otimes T^*M$, then contract the first two factors (as we did to get R from Ric), obtaining a section of T^*M. The equation we get by applying this process to both sides of $d_\nabla F^\nabla = 0$, when F^∇ is the Riemann tensor, is an important identity called the *(doubly) contracted Bianchi identity.* ■

(e) Show that following is equivalent to the Bianchi identity: for all vector fields X, Y, Z on M,

$$(\nabla_X F^\nabla)(Y,Z) + (\nabla_Y F^\nabla)(Z,X) + (\nabla_Z F^\nabla)(X,Y) = 0.$$

Remark 1.7 For a Riemannian manifold (M, g) and a point $p \in M$, some simple algebra shows that the sectional-curvature function $\sigma_p = \sigma|_{G_2(T_pM)}$ is constant if and only if there is a constant c_p such that for all $X, Y, Z \in T_pM$,

$$R(X,Y)Z = c_p \{g(Y,Z)X - g(X,Z)Y\}.$$

(The number c_p is then exactly the constant value of σ_p.) Thus, the function σ_p is constant for *every* point p if and only if for some function $f : M \to \mathbf{R}$, we have

$$R(X,Y)Z = f \{g(Y,Z)X - g(X,Z)Y\}$$

for all vector fields X, Y, Z. Let us say in this case that M has *fiberwise constant sectional curvature.*

If $\dim(M) = 2$, then (M, g) automatically has fiberwise-constant sectional curvature, since for every $p \in M$ the fiber $G_2(T_pM)$ is a single point. But it would appear that if $\dim(M) > 2$, fiberwise-constant sectional curvature is a weaker condition than constant sectional curvature (the latter meaning that the whole function $\sigma : G_2(TM) \to \mathbf{R}$ is constant.) However, the contracted Bianchi identity can be used to show that if M is connected and $\dim(M) > 2$, then *fiberwise*-constant sectional curvature implies *constant* sectional curvature. (Said another way: if the sectional-curvature function is constant on each fiber of the bundle $G_2(TM)$, then it does not even vary from fiber to fiber.)

This is actually a corollary of an even more surprising (and more general) fact. A Riemannian manifold (M, g) is called an *Einstein manifold* if the Ricci tensor is proportional to the metric at each point: $\mathsf{Ric} = fg$ for some $f : M \to \mathbf{R}$. If (M, g) has fiberwise-constant sectional curvature then (M, g) is Einstein, but the converse is false; thus "Einstein" is a more general condition. The contracted Bianchi identity implies that if M is connected and $\dim(M) > 2$, and (M, g) is Einstein, then the function f in "$\mathsf{Ric} = fg$" is constant. ■

7. Torsion and the covariant exterior derivative. Let M be a manifold. The identity map $I : TM \to TM$ may be viewed as a TM-valued 1-form on M. (Note that for a general vector bundle, there is no analog of this special 1-form.)

Let ∇ be a connection on TM.

(a) Show that

$$d_\nabla I = \tau^\nabla, \tag{1.12}$$

where the torsion tensor-field τ^∇ is viewed as a TM-valued 2-form (just as is $d_\nabla I$). I.e. *the torsion of a connection on TM is the covariant exterior derivative of the "identity 1-form" $I \in \Omega^1(M; TM)$.*

Remark 1.8 Above, we treated I as an element of $\Omega^1(M; TM)$; the object $d_\nabla I$ was then an element of $\Omega^2(M; TM)$. But we may also view I as tensor field on M, a section of the bundle $\mathrm{End}(TM) = \mathrm{End}(TM \otimes T^*M)$. (In terms of bundle-valued differential forms, I is then a element of $\Omega^0(M; \mathrm{End}(TM))$ rather than $\Omega^1(M; TM)$.) From the last homework assignment, the connection ∇ on TM induces a connection on $\mathrm{End}(TM)$ (see problems 2cd on the last assignment). With this induced connection, treating I as a section of $\mathrm{End}(TM)$, we have $\nabla I = 0 \in \Gamma(\mathrm{End}(TM) \otimes T^*M) = \Omega^1(M; \mathrm{End}(TM))$. $\blacksquare$

(b) Suppose $\tau^\nabla = 0$. Then, viewing I as a TM-valued 1-form, by part (a) we have $d_\nabla d_\nabla I = d_\nabla(0) = 0 \in \Omega^3(M; TM)$. But by problem 3d(ii), we also have $d_\nabla d_\nabla I = F^\nabla(I) \in \Omega^3(M; TM)$. Hence $F^\nabla(I) = 0$.

In particular "$F^\nabla(I) = 0$" holds if ∇ is the Levi-Civita connection for a Riemannian metric g. Use the definition of $F^\nabla(I)$ (plus the equation $F^\nabla(I) = 0$) to derive a symmetry of the Riemann tensor that we have derived by other means.

8. Connections on the pulled-back tangent bundle. Let $F : N \to M$ be a smooth map of manifolds. As discussed last semester, a vector field on N does not, in general, push forward to a vector field on M. However, it *does* push forward to a section of the pulled-back tangent bundle: Given $X \in \Gamma(TN)$, we can define a section $\hat{X} \in \Gamma(F^*TM)$ by

$$\hat{X}_p := F_p^\sharp(F_{*p}X_p) \tag{1.13}$$

(a) Let ∇' be an arbitrary connection on $F^*(TM)$ (not necessarily pulled back from a connection on TM). Consider the bilinear, antisymmetric "pseudo-torsion" map $\tilde{\tau}_\psi = \tilde{\tau}_\psi^{\nabla'} : \Gamma(TN) \times \Gamma(TN) \to \Gamma(F^*(TM))$ defined by

$$\tilde{\tau}_\psi(X, Y) = \nabla'_X \hat{Y} - \nabla'_Y \hat{X} - \widehat{[X, Y]}.$$

(The subscript ψ is for "pseudo"; there is no object "ψ" here.) Show that $\tilde{\tau}_\psi$ is $\mathcal{F}(N)$-bilinear, hence tensorial, defining a section $\tau_\psi = \tau_\psi^{\nabla'} \in \Omega^2(N; F^*(TM))$.

(b) We may view (1.13) as the definition of a canonical $F^*(TM)$-valued 1-form I_ψ on N,

$$I_\psi(X_p) = \hat{X}_p = F_p^\sharp(F_{*p}X_p).$$

Show that $\tau_\psi = d_{\nabla'} I_\psi$.

(c) Show that if ∇' is the pullback of a connection ∇ on TM whose torsion is $\tau = \tau^\nabla$, then $\tau_\psi^{\nabla'} = F^*\tau$, where we define $F^*\tau$ pointwise by

$$(F_p^*\tau)_p(X_p, Y_p) := F_p^\sharp\left(\tau_{F(p)}(F_{*p}X_p, F_{*p}Y_p)\right), \quad p \in N.$$

Hint: Fix an arbitrary point $p \in N$ and let $\{x^i\}, \{y^i\}$ be local coordinates on a neighborhood of $p, F(p)$ respectively. Compute $\tilde{\tau}_\psi\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$. The Jacobian $\left(\frac{\partial y^i}{\partial x^j}\right)$ will enter your calculation.