$\mathcal{F}$-linearity, tensoriality, and related notions

Contents

1 Some notation and terminology for these notes

- Throughout, M is an arbitrary manifold and $n = \dim(M)$.
- For any vector bundle E over M:

1. $\pi_E : E \to M$ denotes the projection map.
2. For each $p \in M$, $E_p := \pi_E^{-1}(p)$, the fiber of E over p.
3. We use the terminology *set-theoretic section of* E for any map $s : M \to E$, not necessarily smooth (or even continuous), such that $\pi_E \circ s = \mathrm{id}_M$. We reserve the terminology *section of* E for a *smooth* set-theoretic section.
4. $\Gamma(E)$ denotes the space of sections of E.
5. For $s \in \Gamma(E)$, the value of s at p may be denoted $s(p), s_p$, or $s|_p$.
6. For $U \subset M$ open and $s \in \Gamma(E|_U)$, the *extension of* s *by* 0 *to* M is the set-theoretic section $\tilde{s} : M \to E$ such that

$$\tilde{s}(p) = \begin{cases} s(p) & \text{if } p \in U, \\ 0 & \text{if } p \notin U. \end{cases}$$

7. Let $p \in M$, $v \in E_p$, and $s \in \Gamma(E)$. We say that s *is an extension of* v, or that s *extends* v, if $s(p) = v$.
8. If $\kappa = \mathrm{rank}(E) > 0$, then for $U \subset M$ open, a *basis of sections of* E *over* U, or *basis of sections of* $E|_U$, will mean an ordered κ-tuple $\{s_\mu \in \Gamma(E|_U)\}_{\mu=1}^{\kappa}$ such that for all $p \in U$, $\{s_\mu(p)\}$ is a basis of E_p. (This is an abuse of terminology, but is convenient.)

- We write $\mathcal{F} = \mathcal{F}(M) = C^\infty(M)$ (the algebra of smooth functions $M \to \mathbf{R}$). For any vector bundle E over M, there is a natural action of $\mathcal{F}$ on $\Gamma(E)$: for $s \in \Gamma(E)$ and $f \in \mathcal{F}$, we define $fs \in \Gamma(E)$ by $(fs)(p) = f(p)s(p)$. (It is easily seen that the set-theoretic section fs is, indeed, smooth, hence an element of $\Gamma(E)$.) Thus the vector space $\Gamma(E)$ is canonically an $\mathcal{F}$-module.

2 $\mathcal{F}$-linearity and tensoriality

In this section of these notes, E and F denote fixed, arbitrary vector bundles over M.

Definition 2.1 Let $L : \Gamma(E) \to \Gamma(F)$ be a map.

1. We say that L is *$\mathcal{F}$-linear* if $L(s_1 + s_2) = L(s_1) + L(s_2)$ and $L(fs) = fL(s)$ for all $s_1, s_2, s \in \Gamma(E)$ and all $f \in \mathcal{F}$. (Thus every $\mathcal{F}$-linear map is linear.) Equivalent definitions are:
   - L is *$\mathcal{F}$-linear* if L is linear and $L(fs) = fL(s)$ for all $s \in \Gamma(E)$ and all $f \in \mathcal{F}$.
   - An *$\mathcal{F}$-linear map* $\Gamma(E) \to \Gamma(F)$ is a homomorphism of $\mathcal{F}$-modules.

2. We say that L is *tensorial* if there exists a bundle homomorphism $H : E \to F$ (covering the identity map id_M), such that for all $s \in \Gamma(E)$,
$$L(s) = H \circ s. \tag{1}$$
   Correspondingly, if we are given a bundle homomorphism $H : E \to F$, the induced map $L : \Gamma(E) \to \Gamma(F)$ given by (1) will be denoted L_H.

3. For $s \in \Gamma(E)$ and $p \in M$, we say that $L(s)|_p$ *depends only of value of s at p* if for all $s_1 \in \Gamma(E)$ with $s_1(p) = s(p)$, we have $L(s)|_p = L(s_1)|_p$. In these notes, we will say that L *is determined by* 0*-jets* if for all $s \in \Gamma(E)$ and $p \in M$, $L(s)|_p$ depends only of value of s at p.[1]

[1] "Determined by 0-jets" is a phrase invented just for these notes, in order to have a name by which to refer to this property in Proposition 2.8. For any integer $r \geq 0$, there is an object called the *r-jet of a section of E at a point.* We do not define general r-jets in these notes, but the 0-jet of a section $s \in \Gamma(E)$ at a point p is simply the value $s(p) \in E_p$. In a sense that can be made precise, an r-jet of a section s at p captures the "r^{th}-order information" of s at p.

Let $\mathrm{Hom}(E,F)$ denote the vector bundle whose fiber at p is $\mathrm{Hom}(E_p,F_p)$.[2] Observe that there is a natural one-to-one correspondence

$$\begin{aligned}\{\text{bundle homomorphisms } E\to F\} &\longleftrightarrow \Gamma(\mathrm{Hom}(E,F)), &(2)\\ H &\longleftrightarrow \hat{H}. &(3)\end{aligned}$$

Specifically, given a homomorphism $H:E\to F$ and $p\in M$, the map $H|_{E_p}$ is a linear map $\hat{H}_p : E_p \to F_p$, i.e. an element of the fiber $\mathrm{Hom}(E,F)_p := \mathrm{Hom}(E_p,F_p)$. Smoothness of H implies smoothness of $\hat{H}$ (proof left to reader). Thus the map $p\mapsto \hat{H}_p$ is a section of $\mathrm{Hom}(E,F)$. Conversely, given $\hat{H}\in\Gamma(\mathrm{Hom}(E,F))$, we can define a map $H:E\to F$ by $H(v)=\hat{H}_{\pi_E(v)}(v)$ for all $v\in E$. By definition, $\hat{H}_p$ is a (linear) map $E_p\to F_p$ for all p, so $\pi_F(H(v))=\pi_E(v)$. Smoothness of $\hat{H}$ implies smoothness of H (proof left to reader). Thus H is a smooth map $E\to F$ covering the identity map id_M, and linear on fibers. By definition, H is therefore a bundle homomorphism.

For the remainder of these notes, we use the notation (3) for the correspondence (2).

Given H and $\hat{H}$ as above, the canonical isomorphism $\mathrm{Hom}(E,F)\to F\otimes E^*$ identifies the section $\hat{H}$ with a section of $F\otimes E^*$. The action of $\hat{H}$ on a section s (yielding the section $L_H(s)\in\Gamma(F)$) is achieved by pointwise tensor-algebra operations:

$$\begin{array}{ccccccc}\mathrm{Hom}(E,F)_p\times E_p & \to & (F_p\otimes E_p^*)\otimes E_p \underset{\text{canon.}}{\cong} F_p\otimes E_p^*\otimes E_p & \to & F_p & & \\ (\hat{H}_p,s_p) & \mapsto & \hat{H}_p\otimes s_p & \mapsto & \langle \hat{H}_p, s_p\rangle & = & (L_H(s))_p\ ,\end{array} \tag{4}$$

where the last map is *contraction on the last two factors of* $F_p\otimes E_p^*\otimes E_p$ (the linear map $F_p\otimes E_p^*\otimes E_p\to F_p$ induced by the trilinear map $F_p\times E_p^*\times E_p\to F_p$ given by $(w,\alpha,v)\mapsto\langle\alpha,v\rangle w$). This is why we call a map $L:\Gamma(E)\to\Gamma(F)$ *tensorial* if there exists a bundle homomorphism $H:E\to F$ satisfying relation (1).

We will show that for a linear map $\Gamma(E)\to\Gamma(F)$, the notions of $\mathcal{F}$-linearity, tensoriality, and having the property of being determined by 0-jets, are equivalent. We start with some lemmas we will need. Recall that a rank-k vector bundle E' over a manifold M' is called *trivial* if there exists a bundle isomorphism $E'\to M'\times\mathbf{R}^k$.

Lemma 2.2 *Let $p\in M$. There exists a chart (U,ϕ) of M such that $E|_U$ is trivial.*

[2] Since, for any finite-dimensional vector spaces V and W, the space $\mathrm{Hom}(V,W)$ is canonically isomorphic to $W\otimes V^*$, and since $F\otimes E^*$ is a vector bundle over M, we can produce a vector-bundle atlas for $\mathrm{Hom}(E,F)$ by appropriately composing the canonical isomorphisms $\mathrm{Hom}(E_p,F_p)\to F_p\otimes E_p^*$ with vector-bundle-chart maps for $F\otimes E^*$. This ensures that $\mathrm{Hom}(E,F)$ is indeed a vector bundle, and not *just* the disjoint union of the vector spaces $\mathrm{Hom}(E_p,F_p)$.

Proof: Let (U_1, ϕ) be a chart of M with $p \in U_1$. Let V be an open neighborhood of p such that $E|_V$ is trivial. Let $U = U_1 \bigcap V$, $\phi = \phi_1|_U$. Then (U, ϕ) is a chart with the desired property. ■

Lemma 2.3 *Let $p \in M$, let (U, ϕ) be a chart of M such that $E|_U$ is trivial, let $B \subset \mathbf{R}^n$ be an open ball with $\overline{B} \subset \phi(U)$, and let $V = \phi^{-1}(B)$. Suppose $s \in \Gamma(E)$ is a section supported in $\overline{V}$ (i.e. identically zero on the complement). Assume that $\kappa := \mathrm{rank}(E) > 0$. Then there exist a κ-tuple $\{t_\mu \in \Gamma(E)\}_{\mu=1}^{\kappa}$ and a κ-tuple $\{h^\mu \in C^\infty(M)\}_{\mu=1}^{\kappa}$ such that (i) $\{t_\mu|_V\}_{\mu=1}^{\kappa}$ is a basis of sections of $E|_V$, and (ii) $s = \sum_\mu h^\mu t_\mu$.*

The point of this lemma is to show that any $s \in \Gamma(E)$ can be expressed *globally* as a "linear" combination, with coefficients in $\mathcal{F}$, of *global* sections of E that restrict to a basis of sections of $E|_V$.

Proof of Lemma 2.3: Let $\{s_\mu\}_{\mu=1}^{\kappa}$ be a basis of sections of $E|_U$. Let $\rho : M \to \mathbf{R}$ be a smooth function such that $\rho \equiv 1$ on V and $\rho \equiv 0$ on $M \setminus U$.

Since $\{s_\mu\}$ is a basis of sections of $E|_U$, there exist unique smooth functions $f^\mu : U \to \mathbf{R}$ such that $s|_U = \sum_{\mu=1}^{\kappa} f^\mu s_\mu$. Since $\mathrm{supp}(s) \subset \overline{V} \subset U$, and the s_μ are linearly independent at each point of U, we have $\mathrm{supp}(f^\mu) \subset \overline{V}$ for each μ. For each μ the function $\rho|_U f^\mu$ is smooth and supported in $\overline{V} \subset U$, hence extends smoothly by 0 to a function $h^\mu : M \to \mathbf{R}$ (still supported in $\overline{V}$). Similarly, the section $\rho|_U s_\mu$ is smooth and supported in U, hence extends smoothly by 0 to a section t_μ of E, supported in U.

Let $\tilde{s} = \sum_{\mu=1}^{\kappa} h^\mu t_\mu$. Then for $p \in M \setminus V$, we have $\tilde{s}(p) = 0 = s(p)$, since s and the h^μ are supported in $\overline{V}$. For $p \in V$, we have $\rho(p) = 1$, implying $h^\mu(p) = f^\mu(p)$ and $t_\mu(p) = s_\mu(p)$, hence implying $\tilde{s}(p) = s(p)$.

Therefore, we have the global equalities $s = \tilde{s} = \sum h^\mu t_\mu$. ■

Lemma 2.4 *Let $L : \Gamma(E) \to \Gamma(F)$ be $\mathcal{F}$-linear. Let $p \in M$, $s \in \Gamma(E)$, and assume that $s(p) = 0$. Then $L(s)|_p = 0$.*

Proof: It suffices to assume that $\kappa := \mathrm{rank}(E) > 0$. Let (U, ϕ) be a chart of M such that $E|_U$ is trivial and $p \in U$. Let $B = B_r(\phi(p)) \subset \mathbf{R}^n$ be the open ball of radius r centered at $\phi(p) \in B$, with r small enough that and $\overline{B} \subset \phi(U)$. Let $B_1 = B_{r/2}(\phi(p))$, $V = \phi^{-1}(B)$, and $V_1 = \phi^{-1}(B_1)$. Let $\rho : M \to \mathbf{R}$ be a smooth function such that $\rho \equiv 1$ on V_1 and $\rho \equiv 0$ on $M \setminus V$. Let $s_1 = \rho s$ and $s_2 = (1 - \rho)s$; thus $s = s_1 + s_2$.

Since $\mathrm{supp}(s_1) \subset V$, Lemma 2.3 implies that we can write $s_1 = \sum_{\mu=1}^{\kappa} h^\mu t_\mu$ for some functions $h^\mu \in C^\infty(M)$ and some sections $t_\mu \in \Gamma(E)$ such that $\{t_\mu\}_1^\kappa$ is a basis of sections of $E|_V$. Observe that $s_1(p) = 0$. Since $\{t_\mu(p)\}$ is a basis of E_p, it follows that $h^\mu(p) = 0$ for each μ. The $\mathcal{F}$-linearity of L implies that $L(s_1) = \sum h^\mu L(t_\mu)$. Hence for all $p \in M$, $L(s_1)_p = \sum h^\mu(p) L(t_\mu)|_p = 0$.

Again using $\mathcal{F}$-linearity, $L(s_2)|_p = (1 - \rho(p))L(s)|_p = 0$, since $\rho(p) = 1$. Hence $L(s)|_p = L(s_1)|_p + L(s_2)|_p = 0$. ■

Corollary 2.5 *Let $L : \Gamma(E) \to \Gamma(F)$ be $\mathcal{F}$-linear. Then L is determined by 0-jets.*

Proof: Let $p \in M$, let $s_1, s_2 \in \Gamma(E)$, and assume that $s_1(p) = s_2(p)$. Let $s = s_2 - s_1$. Then $s(p) = 0$, so by Lemma 2.4, $L(s)|_p = 0$. Hence $L(s_2)|_p = L(s + s_1)|_p = L(s_1)|_p$. ■

Lemma 2.6 (Extendability of sections defined at a point) *Let $p \in M$, $v \in E_p$. There exists $s \in \Gamma(E)$ that extends v.*

Proof: Let V be an open neighborhood of p such that $E|_V$ is trivial, and let $\{s_\mu\}_1^\kappa$ be a basis of sections of $E|_V$. Let $\{c^\mu \in \mathbf{R}\}_{\mu=1}^\kappa$ be such that $v = \sum c^\mu s_\mu(p)$.

Let $\rho : M \to \mathbf{R}$ be a smooth function such that $\rho(p) = 1$ and $\operatorname{supp}(\rho) \subset V$. Define $s' \in \Gamma(E|_V)$ by $s' = \rho \sum c^\mu s_\mu$; thus $s'(p) = v$. Let s be the extension of s' by 0 to M. Then s is smooth, hence a section of E, and $s(p) = v$. ■

Corollary 2.7 *If $L : \Gamma(E) \to \Gamma(F)$ is tensorial, then the bundle homomorphism H in (1) is unique.*

Proof: Let H_1, H_2 be bundle homorphisms $E \to F$ satisfying (1). Let $H' = H_2 - H_1$ (defined pointwise). Then $(H_2 - H_1)(s(p)) = 0$ for all $p \in M, s \in \Gamma(E)$. Since for all $v \in E$, there exists a section $s \in \Gamma(E)$ extending v, it follows that $(H_2 - H_1)(v) = 0$ for all $v \in E$. Hence $H_2 = H_1$. ■

Proposition 2.8 *Let $L : \Gamma(E) \to \Gamma(F)$ be a linear map. Then the following are equivalent:*

(i) L *is* $\mathcal{F}$*-linear.*

(ii) L *is determined by* 0*-jets.*

(iii) L *is tensorial.*

Proof: We show "(iii) $\implies$ (i) $\implies$ (ii) $\implies$ (iii)."

(iii) $\implies$ (i): This follows immediately from (1) and the definition of "bundle homomorphism covering the identity".

(i) $\implies$ (ii): This is Corollary 2.5.

(ii) $\implies$ (iii): Assume that L is determined by 0-jets. For $p \in M$ and $v \in E_p$, define $\hat{H}_p(v) \in F_p$ by

$$\hat{H}_p(v) := L(s)|_p, \tag{5}$$

where s is any extension of v to a section of E (such an extension exists by Lemma 2.6). Since L is determined by 0-jets, $\hat{H}_p(v)$ is well-defined; all extensions s of v yield the same value of $L(s)|_p$. Letting v vary over E_p, (5) therefore defines a map $\hat{H}_p : E_p \to F_p$.

If s_1, s_2 are extensions of $v_1, v_2 \in E_p$ to sections of E, and $c_1, c_2 \in \mathbf{R}$, then then $c_1s_1 + c_2s_2$ is an extension of $c_1v_1 + c_2v_2$ to a section of E. The linearity of L therefore implies that, for each p, the map $\hat{H}_p : E_p \to F_p$ is linear. Hence, letting p vary, we obtain a set-theoretic section $\hat{H}$ of $\mathrm{Hom}(E, F)$.

We next show that $\hat{H}$ is smooth. Let $p \in M$, let U be a neighborhood of p such that $E|_U$ and $F|_U$ are trivial, and let $\{s_\mu\}_{\mu=1}^{\kappa_1}, \{\sigma_\nu\}_{\nu=1}^{\kappa_2}$, be bases of sections of $E|_U, F|_U$ respectively. Let $\{\xi^\nu\}_{\nu=1}^{\kappa_2}$ be the basis of sections of $F^*|_U$ that is dual (pointwise) to $\{\sigma_\nu\}_{\nu=1}^{\kappa_2}$. Let $A : U \to \{\kappa_2 \times \kappa_1 \text{ matrices}\}$ be the function defined pointwise by expanding the elements $\hat{H}_q(s_\mu(q)) \in F_q$ in terms of the basis $\{\sigma_\nu(q)\}$ of F_q:

$$\hat{H}_q(s_\mu(q)) = \sum_{\nu=1}^{\kappa_2} \sigma_\nu(q) A^\nu{}_\mu(q), \quad q \in U, \quad 1 \le \mu \le \kappa_1 \ .$$

Alternatively, $A^\nu{}_\mu(q) = \langle \xi^\nu|_q, \hat{H}(s_\mu(q))\rangle$. To show that $\hat{H}$ is smooth at p, it suffices to show that the each coefficient-function $A^\nu{}_\mu$ is smooth at p.

Let $\rho : M \to \mathbf{R}$ be a smooth function such that $\rho \equiv 1$ on some open neighborhood V of p and $\rho \equiv 0$ on $M \setminus U$. The sections $\rho|_U s_\mu$ of $E|_U$ extend smoothly by 0 to sections t_μ of E, and we have $t_\mu(q) = s_\mu(q)$ for all $q \in V$. Hence for $q \in V$,

$$A^\nu{}_\mu(q) = \langle \xi^\nu|_q, \hat{H}(s_\mu(q))\rangle = \langle \xi^\nu|_q, \hat{H}(t_\mu(q))\rangle = \langle \xi^\nu|_q, L(t_\mu)|_q\rangle \tag{6}$$

Since $t_\mu \in \Gamma(E)$, $L(t_\mu)$ is a (smooth) section of F. Hence both ξ^ν and $L(t_\mu)$ are smooth on V, so (6) implies that the functions $A^\nu{}_\mu$ are smooth on V. In particular, they are smooth at p.

Thus $\hat{H}$ is smooth at p. Since p was arbitrary, $\hat{H} \in \Gamma(E)$. Using the correspondence (2)–(3), we obtain a bundle homomorphism $H : E \to F$ such that (1) holds. Hence L is tensorial. ■

Notation 2.9 For vector bundles E, F over M, let $\mathrm{Hom}_{\mathcal{F}}(\Gamma(E), \Gamma(F))$ denote the set of $\mathcal{F}$-linear maps $\Gamma(E) \to \Gamma(F)$.

Remark 2.10 For $L \in \mathrm{Hom}_{\mathcal{F}}(\Gamma(E), \Gamma(F))$ let us write H_L for what Corollary 2.7 t guarantees us is the *unique* bundle homomorphism H for which $L = L_H$. (Thus $H = H_L \iff L = L_H$.) Then, using (2)–(3), we obtain a natural map

$$\begin{aligned} j_F^E : \mathrm{Hom}_{\mathcal{F}}(\Gamma(E), \Gamma(F)) &\to \Gamma(\mathrm{Hom}(E, F)), \\ L &\mapsto \hat{H}_L := \widehat{(H_L)}\ . \end{aligned} \tag{7}$$

Observe that $\mathrm{Hom}_{\mathcal{F}}(\Gamma(E), \Gamma(F))$ is a vector space—a subspace of $\mathrm{Hom}(\Gamma(E), \Gamma(F))$ —and, by Proposition (2.8), is precisely the space of tensorial maps $\Gamma(E) \to \Gamma(F)$. Furthermore, the space $\mathrm{Hom}_{\mathcal{F}}(\Gamma(E), \Gamma(F))$ is itself an $\mathcal{F}$-module, and it is easily seen that j_F^E is an $\mathcal{F}$-module isomorphism.

Remark 2.11 A map $L : \Gamma(E) \to \Gamma(F)$ is called *local* if, for all $p \in M$ and $s \in \Gamma(E)$, the value of $L(s)|_p$ depends only on the germ of s at p.[3] (If L is linear, then L is local if and only if for every open set $U \subset M$ and every $s \in \Gamma(E)$ such that $s|_U \equiv 0$, we have $L(s)|_U \equiv 0$.) Obviously, if L depends only on 0-jets, then L is local, but the converse is very far from true. Contained in the set of all local maps[4] $\Gamma(E) \to \Gamma(F)$ is the set of all *differential operators* $\Gamma(E) \to \Gamma(F)$. A *differential operator* $\Gamma(E) \to \Gamma(F)$ *of order* 0 is, by definition, a map that is determined by 0-jets. In these notes we do not define what "differential operator $\Gamma(E) \to \Gamma(F)$" means in general, but a true fact is that for each integer $r \geq 0$ there is a notion of *differential operator of order* r that, when E and F are product bundles, reduces to exactly what one would expect. As one might expect from the name "differential operator", and the local nature of anything that we generally call "differentiation", any differential operator of any order is local.

[3]Germs of arbitrary maps $f : X \to Y$, where X is a topological space, are defined analogously to germs of real-valued functions on a manifold. For each $p \in X$, let $\mathfrak{N}(p)$ denote the set of open neighborhoods of p, and define a relation $\sim_p$ on the set $\{(U, g) : U \in \mathfrak{N}(p) \text{ and } g : U \to Y \text{ is a function}\}$ by declaring $(U_1, g_1) \sim_p (U_2, g_2) \iff U_1 \cap U_2$ contains some $V \in \mathfrak{N}(p)$ for which $g_1|_V = g_2|_V$. The *germ at* p of $f : X \to Y$ is then defined to be the equivalence class of the pair (X, f) under the relation $\sim_p$.

[4]The term "local map" is being used here with the very specific meaning above. Outside this context, the same term could be used to mean something very different, e.g. a map defined just on some open subset of a topological space.

3 $\mathcal{F}$-multilinearity and tensoriality

For *any* vector spaces V, W, finite- or infinite-dimensional, we write $\mathrm{Hom}(V, W)$ for the space of all linear maps $V \to W$. In this notation, we do not care if the vector spaces are topologized, let alone whether our linear maps are continuous.

Definition 3.1 Let $E_1, E_2, \ldots, E_r, F$ be vector bundles over M, and let $L : \Gamma(E_1) \times \Gamma(E_2) \times \cdots \times \Gamma(E_r) \to \Gamma(F)$ be a map.

1. We say that L is *$\mathcal{F}$-multilinear* if for $1 \leq i \leq r$, L is $\mathcal{F}$-linear as a function of its i^{th} argument with the other arguments held fixed.

2. We say that L is *tensorial* if there exists a bundle homomorphism $H : E_1 \otimes E_2 \ldots \otimes E_r \to F$, covering the identity, such that for all $s_i \in \Gamma(E_i)$, $1 \leq i \leq r$,
$$L(s_1, s_2, \ldots, s_r) = H \circ (s_1 \otimes s_2 \ldots \otimes s_r). \tag{8}$$
Here $s_1 \otimes s_2 \ldots \otimes s_r$ is the section of $E_1 \otimes E_2 \ldots \otimes E_r$ defined by pointwise tensor-product:
$$\begin{array}{rcl} (s_1 \otimes s_2 \ldots \otimes s_r)|_p = s_1(p) \otimes s_2(p) \ldots \otimes s_r(p) & \in & E_1|_p \otimes E_2|_p \ldots \otimes E_r|_p \\ & \underset{\text{canon.}}{\cong} & (E_1 \otimes E_2 \ldots \otimes E_r)_p \ . \end{array}$$

3. For $s_i \in \Gamma(E_i), 1 \leq i \leq r$, and $p \in M$, we say that $L(s_1, \ldots, s_r)|_p$ *depends only of values of* $s_1, \ldots, s_r$ *at* p if for all $s_i' \in \Gamma(E_i)$ with $s_i'(p) = s_i(p)$, $1 \leq i \leq r$, we have $L(s_1', \ldots, s_r')|_p = L(s_1, \ldots, s_r)|_p$. In these notes, we will say that L *is determined by* 0*-jets* if for all $s_i \in \Gamma(E_i), 1 \leq i \leq r$, and $p \in M$, the value $L(s_1, \ldots, s_r)|_p$ depends only of values of $s_1, \ldots, s_r$ at p.

Note that from (2)–(3), we have a natural one-to-one correspondence

$$\begin{aligned} &\{\text{bundle homomorphisms } E_1 \otimes E_2 \otimes \ldots \otimes E_r \to F\} \\ &\quad \longleftrightarrow \Gamma(\mathrm{Hom}(E_1 \otimes E_2 \otimes \ldots \otimes E_r, F)), \end{aligned} \tag{9}$$

which we will again denote by $H \longleftrightarrow \hat{H}$.

For any vector spaces $V_1, V_2, \ldots, V_r, Z$, let $\mathrm{Multihom}(V_1, \ldots, V_r, Z)$ denote set of multilinear maps $V_1 \times V_2 \times \cdots \times V_r \to Z$. (When $r = 1$, "Multihom" means the same thing as "Hom". For other small values of r we may replace the "Multi" in "Multihom" by a more specific prefix, as in $\mathrm{Bihom}(V_1 \times V_2, Z)$[5] and

[5]In some other notes, and in class in spring 2022, I have used the notation $\mathrm{Bil}(V \times W, Z)$ for $\mathrm{Bihom}(V \times W, Z)$, and have used the notation $Z^{V \times W}$ for $\mathrm{Maps}(V \times W, Z)$ (at least when $Z = \mathbf{R}$).

Trihom($V_1 \times V_2 \times V_3, Z$).) This set is a vector subspace of Maps($V_1 \times \cdots \times V_r, Z$), the space of *all* functions $V_1 \times \cdots \times V_r \to Z$ (with vector-space operations defined pointwise: $(f_1 + f_2)(p) = f_1(p) + f_2(p)$, etc.).

Recall that for any nonempty *sets* $X_1, \ldots X_r, Z$, where $r \geq 2$, the natural map

$$\begin{array}{rcl} \mathrm{Maps}(X_1 \times \cdots \times X_r, Z) & \stackrel{\natural}{\to} & \mathrm{Maps}(X_1, \mathrm{Maps}(X_2 \times \cdots \times X_r, Z)), \\ f & \mapsto & f_\natural : x_1 \mapsto f(x_1, \cdot) \end{array} \tag{10}$$

is a one-to-one correspondence. (In (10), $f_\natural(x)_1 = f(x_1, \cdot) \in \mathrm{Maps}(X_2 \times \cdots \times X_r, Z))$ is the map that sends $(x_2, \ldots, x_r)$ to $f(x_1, x_2, \ldots, x_r)$.) For vector spaces $V_1, \ldots, V_r, Z$, one can easily verify that $\natural$ restricts to an isomorphism

$$\natural_{\mathbf{R}} : \mathrm{Multihom}(V_1 \times \cdots \times V_r, Z) \to \mathrm{Hom}(V_1, \mathrm{Multihom}(V_2 \times \cdots \times V_r, Z)).$$

Proposition 2.8 generalizes to the ($\mathcal{F}$-)multilinear setting:

Proposition 3.2 *Let $E_1, \ldots E_r, F$ be vector bundles over M and let $L : \Gamma(E_1) \times \cdots \times \Gamma(E_r) \to \Gamma(F)$ be a multilinear map. Then the following are equivalent:*

(i) L is $\mathcal{F}$-multilinear.

(ii) L is determined by 0-jets.

(iii) L is tensorial.

Proof: First suppose $r = 2$. We will show "(iii) $\implies$ (i) $\implies$ (ii) $\implies$ (iii)."

(iii) $\implies$ (i): This follows immediately from (8) in Definition 3.1.

(i) $\implies$ (ii): Assume L is $\mathcal{F}$-bilinear. Then for fixed $s_1 \in \Gamma(E_1)$, the map $\Gamma(E_2) \to \Gamma(F), s_2 \mapsto L(s_1, s_2)$, is $\mathcal{F}$-linear. Hence by Corollary 2.5, holding s_1 fixed, for $p \in M$ the value $L(s_1, s_2)|_p$ depends only on the value of s_2 at p. Similarly, with s_2 held fixed, the value $L(s_1, s_2)|_p$ depends only on the value of s_1 at p. Hence given $p \in M$, and sections s_1, s_1' of E_1, s_2, s_2' of E_2, such that $s_i(p) = s_i'(p)$ for $i = 1, 2$, we have $L(s_1', s_2')|_p = L(s_1', s_2)|_p = L(s_1, s_2)|_p$. Hence $L(s_1, s_2)|_p$ depends only on the values of s_1 and s_2 at p. Thus L is determined by 0-jets.

(ii) $\implies$ (iii): Assume that L is determined by 0-jets. Let $L' = \natural_{\mathbf{R}}(L) \in \mathrm{Hom}(\Gamma(E_1), \mathrm{Hom}(\Gamma(E_2), \Gamma(F)))$. Fix $s_1 \in \Gamma(E_1)$. Then $L'(s_1) \in \mathrm{Hom}(\Gamma(E_2), \Gamma(F))$ depends only on 0-jets. Hence, by Proposition 2.8, $L'(s_1)$ is $\mathcal{F}$-linear and tensorial, so there exists a bundle homomorphism $H^{(s_1)} : E_2 \to F$ such that $L(s_1, s_2) = L_{H^{(s_1)}}(s_2)$.

Letting s_1 vary, we now have a map $L'' : \Gamma(E_1) \to \Gamma(\mathrm{Hom}(E_2, F))$, $s_1 \mapsto \widehat{H}^{(s_1)}$. Since L is linear, so is L'', and since L is determined by 0-jets, so is L''. Hence, using Proposition 2.8 again, L'' is tensorial, so there exists $\widehat{H} \in \Gamma(\mathrm{Hom}(E_1, \mathrm{Hom}(E_2, F)))$ such that for all $p \in M$, $L(s_1, s_2)|_p = (L'(s_1))(s_2)\big|_p = \left(\widehat{H}_p(s_1(p))\right)(s_2(p))$. But using the canonical isomorphisms

$$\begin{aligned}
\mathrm{Hom}(E_1, \mathrm{Hom}(E_2, F)) &\underset{\text{canon.}}{\cong} \mathrm{Hom}(E_2, F) \otimes E_1^* \\
&\underset{\text{canon.}}{\cong} F \otimes E_2^* \otimes E_1^* \\
&\underset{\text{canon.}}{\cong} F \otimes E_1^* \otimes E_2^* \\
&\underset{\text{canon.}}{\cong} F \otimes (E_1 \otimes E_2)^* \\
&\underset{\text{canon.}}{\cong} \mathrm{Hom}(E_1 \otimes E_2, F),
\end{aligned}$$

we can canonically identify $\hat{H}$ with a section of $\mathrm{Hom}(E_1 \otimes E_2, F)$. It follows from the correspondence (9) that L is tensorial.

The general-r case follows from similar arguments and induction. ■