Differential Geometry—MTG 6257—Spring 2022
Problem Set 1
Due-date: Wednesday, Feb. 2

Required reading: all the problems. Some problems introduce terminology and ideas you'll need in later problems.

Required problems (to be handed in): 1a, 3c, 4, 5bef, 6cd, 7f. In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Optional problems: All the ones that are not required.

1. **Extension from a closed submanifold.** This problem is another valuable application of partitions of unity. You should find the arguments for all three parts very similar to each other.

Let M be a manifold, $Z \subset M$ a submanifold that is closed as a subset of M.

(a) "*Smooth Tietze Extension Theorem*". Suppose $f : Z \to \mathbf{R}$ is a smooth function. Show that f can be extended to a smooth function $M \to \mathbf{R}$.

Note: This would be false without the hypothesis that Z is closed in M, even if we were looking just for *continuous* extensions, and even if we required $\dim(Z)$ to be strictly smaller than $\dim(M)$. (Example: $M = S^2$, $Z =$ equator $\setminus$ {one point}.) If your argument doesn't use the hypothesis that Z is closed, you've made a mistake. The same goes for parts (b) and (c).

(b) A *vector field along* Z is a section of $TM|_Z$, i.e. a smooth map $X : Z \to TM$, $p \mapsto X_p \in T_pM$. (We do not require X_p to be tangent to Z.) Show that a vector field along Z can be extended to a vector field on M.

(c) Similarly, for $k > 0$ a *k-form along* Z is a map $\omega : Z \to \bigwedge^k T^*M$, $p \mapsto \omega_p \in \bigwedge^k T_p^*M$, smooth in the sense that if $X_1, \ldots X_k$ are smooth vector fields along Z, then $p \mapsto \omega(X_1, \ldots, X_k)\big|_p$ is smooth. Show that a k-form along Z can be extended to a k-form on M.

2. Recall that a topological space X is *arcwise connected* (or *path-connected*) if for all $p, q \in X$, there exists a continuous map $\gamma : [0, 1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$. It is easily shown that every arcwise connected space is connected (a separation of X would lead to a separation of $[0, 1]$), but there are connected spaces that are not arcwise connected (the famous example being the "topologist's sine curve").

Show that a manifold M is connected if and only if M is arcwise connected. (You may assume the "arcwise connected $\implies$ connected" half of this iff; you need only show the "connected implies arcwise connected" half.)

Note: This problem was inserted here because its result can be used to simplify arguments in later problem-parts involving connectedness. However, most of those problem-parts can be done without any reliance on arcwise-connectedness.

3. Let $n \geq 1$, let M and N be oriented n-dimensional manifolds, and let $F : N \to M$ be a smooth map. Recall that at any point of M or N, a basis of the tangent space is called either *positively oriented* or *negatively oriented*, according to whether basis is or is not in the orientation class defined that manifold's given orientation.

(a) Let $p \in N$ and suppose that the derivative $F_{*p} : T_pN \to T_{F(p)}M$ is an isomorphism. Show that if F_{*p} carries *some* positively oriented basis of T_pN to a positively oriented basis of $T_{F(p)}M$, then F_{*p} carries *every* positively oriented basis of T_pN to a positively oriented basis of $T_{F(p)}M$. Similarly, show that if F_{*p} carries *some* positively oriented basis of T_pN to a negatively oriented basis of $T_{F(p)}M$, then F_{*p} does that *every* positively oriented basis of T_pN.

Part (a) shows that the following definition is unambiguous.

> **Definition.** For a given $p \in N$, we say that F is *orientation-preserving at p* (respectively, *orientation-reversing at p*) if F_{*p} carries positively oriented bases of T_pN to positively (respectively, negatively) oriented bases of $T_{F(p)}M$. We say that F is *orientation-preserving* (respectively, *orientation-reversing*) if F is orientation-preserving at every $p \in M$ (respectively, orientation-reversing at every $p \in M$).

Note that for F to be either orientation-preserving or orientation-reversing at a point p, the map F_{*p} must be an isomorphism. Hence the only maps $N \to M$ that can *possibly* be orientation-preserving or orientiation-reversing (globally) are local diffeomorphisms.

(b) For any $p \in N$ or $q \in M$, recall that the given manifold-orientations also define what we mean by *positive* and *negative* elements of the 1-dimensional vector space $\bigwedge^n T_p^*N$ or $\bigwedge^n T_q^*M$. Show that F is orientation-preserving at $p \in N$ (respectively, orientation-reversing at $p \in N$) if and only if the pullback map $F^* : \bigwedge^n T_{F(p)}^*M \to \bigwedge^n T_p^*N$ carries some, and hence any, positive element of $\bigwedge^n T_{F(p)}^*M$ to a positive (respectively, negative) element of $\bigwedge^n T_p^*N$.

(c) Assume that N is connected and that $F : N \to M$ is a diffeomorphism. (i) Show that F is either orientation-preserving or orientation-reversing. (ii) Let $\omega \in \Omega_c^n(M)$ (the space of n-forms of compact support). Show that $F^*\omega$ has compact support (ensuring that $\int_N F^*\omega$ is defined), and prove the following:

$$\int_N F^*\omega = \pm \int_M \omega,$$

with the plus sign if F preserves orientation, and the minus sign if F reverses orientation. (This fact is called *invariance of the integral under diffeomorphism.*)

4. Let M and $\widetilde{M}$ be manifolds of equal dimension, and assume that $F : \widetilde{M} \to M$ is a submersion. Note that, for dimensional reasons and a homework problem from last semester, "F is a submersion" is equivalent to F being a local diffeomorphism.

(a) Show that an orientation of M (if one exists) induces, via F, an orientation on $\widetilde{M}$. (Hence if M is orientable, so is $\widetilde{M}$.)

For the remaining parts of this problem, assume that $\widetilde{M}$ is compact and that M is connected. Since F is already assumed to be a submersion, a homework problem from last semester shows that F is surjective. Hence $M = F(\widetilde{M})$ is compact as well.

(b) Prove that F is a smooth covering map; i.e. that for all $p \in M$ there exists an open neighborhood U of p such that $F^{-1}(U)$ is a disjoint union of sets $\tilde{U}_i$ for which $F|_{\tilde{U}_i} : \tilde{U}_i \to U$ is a diffeomorphism. (Here i runs over some index set $\mathcal{I}(p)$, possibly depending on p.)

(c) Prove that for all $p \in M$, the set $F^{-1}(p)$ is finite. (Recall that "$F^{-1}(p)$" is common but imprecise notation for $F^{-1}(\{p\})$.)

(d) Prove that the cardinality of the finite set $F^{-1}(p)$ is independent of p. This finite common value—the number of points in the pre-image of any $p \in M$—is called the *degree* of F as a covering map.[1] (More generally, we may use this definition of degree of a covering map F any time the cardinality of $F^{-1}(p)$ is independent of p, whether or not $\widetilde{M}$ is compact or M is connected.)

(e) Assume that M is oriented, and give $\widetilde{M}$ the induced orientation. Show that for all $\omega \in \Omega^n(M)$,

$$\int_{\widetilde{M}} F^*\omega = (\deg F) \int_M \omega.$$

(The compactness of $\widetilde{M}$ and M ensures that both integrals are defined.)

5. Let M be an n-dimensional manifold, $n \geq 1$. We can construct a manifold called the *orientation double-cover* $\widetilde{M}$ of M as follows. For each $p \in M$ let $\mathrm{Orn}(p)$ denote the set of orientations of T_pM, a two-element set. Given $\sigma \in \mathrm{Orn}(p)$, we let $-\sigma$ denote the other orientation. As a set, let $\widetilde{M} = \bigcup_{p \in M} \mathrm{Orn}(p)$. There is a natural two-to-one map $\pi : \widetilde{M} \to M$ carrying both elements of $\mathrm{Orn}(p)$ to p. We give $\widetilde{M}$ the

[1] In this coarse usage of the word "degree" for covering maps, the degree is always positive. For more general maps between compact, oriented manifolds of equal dimension, there is a notion of degree in which the degree can be positive, negative, or zero. For example, if $\widetilde{M} = M = S^1$ = unit circle in $\mathbf{C}$, for $0 \neq n \in \mathbf{Z}$ the degree of the map $z \mapsto z^n$, as defined in this problem, is $|n|$. But for these maps it makes sense to refine the definition of degree, and even include the case $n = 0$, declaring the degree of $z \to z^n$ to be n whether this integer is positive, negative, or zero. This refined degree then classifies homotopy classes of maps $S^1 \to S^1$; every continuous map is homotopic to $z \mapsto z^n$ for a unique $n \in \mathbf{Z}$.

topology induced by the map π (i.e. a set $\tilde{U} \subset \widetilde{M}$ is declared to be open if and only if $\pi(\tilde{U})$ is open).

It can be shown that every manifold has as an atlas $\{(U_\alpha, \phi_\alpha)\}$ for which all the sets U_α and nonempty intersections $U_\alpha \bigcap U_\beta$ are connected[2]. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ be such an atlas for M. Then, for each $\alpha \in A$, the set $\pi^{-1}(U_\alpha)$ has two connected components, which are distinguished from each other as follows. For $p \in U_\alpha$ let $\sigma_\alpha(p)$ be the orientation of T_pM pulled back by the map $\phi_\alpha : U_\alpha \to \mathbf{R}^n$, where $\mathbf{R}^n$ has its standard orientation. Each $\tilde{p} \in \pi^{-1}(U_\alpha)$ is, by definition, an orientation of $T_{\pi(p)}M$; hence $\tilde{p} = \pm\sigma_\alpha(\pi(\tilde{p}))$ (where "$+\sigma$" means σ). The sign in this formula is constant on each connected component of $\pi^{-1}(U_\alpha)$ (why?). We define $\tilde{U}_{\alpha,+}$ to be the component on which $\tilde{p} = \sigma_\alpha(\pi(\tilde{p}))$, and $\tilde{U}_{\alpha,-}$ to be the component on which $\tilde{p} = -\sigma_\alpha(\pi(\tilde{p}))$. We define corresponding chart-maps $\tilde{\phi}_{\alpha,\pm} : \tilde{U}_{\alpha,\pm} \to \mathbf{R}^n$ as follows. Let $r : \mathbf{R}^n \to \mathbf{R}^n$ be the reflection $(x^1, x^2, \ldots x^n) \mapsto (-x^1, x^2, \ldots, x^n)$. Then we define $\tilde{\phi}_{\alpha,+} = \phi_\alpha \circ \pi$, $\tilde{\phi}_{\alpha,-} = r \circ \phi_\alpha \circ \pi$.

(a) Let $\tilde{A} = A \times \{+, -\}$, an index set for the pairs $(\tilde{U}_{\alpha,\pm}, \tilde{\phi}_{\alpha,\pm})$ constructed above. Show that $\{\tilde{U}_{\tilde{\alpha}}, \tilde{\phi}_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \tilde{A}}$ is an atlas for $\widetilde{M}$, hence that $\widetilde{M}$ is a manifold. (You may assume that paracompactness and Hausdorffness of M imply that $\widetilde{M}$ has these properties. This is not hard to show, but your time would be better spent on other problems in this assignment.)

(b) Show that the atlas $\{\tilde{U}_{\tilde{\alpha}}, \tilde{\phi}_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \tilde{A}}$ is oriented (whether or not M is orientable!). Hence $\widetilde{M}$ is orientable; even better, the construction above gives it a *canonical orientation*, the one induced by this atlas. (It can be shown that this orientation is independent of the atlas of M that we started with, but I'm not asking you to show that.)

(c) Show that $\pi : \widetilde{M} \to M$ is a (smooth), degree-two covering map.

Discussion to set up part (d). From the definition of "covering map", it is easily shown that $\widetilde{M}$ has the following "path-lifting property": given any continuous map $\gamma : [0,1] \to M$, and any $\tilde{p} \in \pi^{-1}(\gamma(0))$, there exists a unique continuous map $\tilde{\gamma} : [0,1] \to \widetilde{M}$ with $\tilde{\gamma}(0) = \tilde{p}$. (You may assume this, but you should be able to prove it on your own, using nothing about manifolds other than that they are topological spaces. The degree of the cover, or whether the cover even has finite degree, is also irrelevant. The same argument works just as easily for any covering space of any topological space.) For general covering spaces, such a curve $\tilde{\gamma}$ is called a *lift* of γ; in the context of the orientation double-cover $\widetilde{M} \xrightarrow{\pi} M$ we may call such a lift an "orientation of M along γ".

[2]It takes some non-trivial work to show this. Just assume it's true for now.

(d) Assume M is connected. Show that M is orientable if and only if $\widetilde{M}$ is *not* connected. (Thus, if we start with a non-orientable, connected M, we obtain a counterexample to the [false] converse of the parenthetic conclusion of problem 4(b).) For the case in which M is orientable, show that $\widetilde{M}$ is diffeomorphic to $M \times \mathbf{Z}_2$, the disjoint union of two "copies" of M.

(e) Since every point in $\widetilde{M}$ is an orientation of a vector space, there is a natural map $\tau : \widetilde{M} \to \widetilde{M}$ defined by $\tau(\sigma) = -\sigma$ (this map is called an *involution*, a term you may recall from group theory, because $\tau \circ \tau$ is the identity map but τ itself is not the identity map). Show that τ is an orientation-reversing map.

(f) Since $\widetilde{M}$ is oriented, we may integrate any compactly supported n-form over $\widetilde{M}$. Show that if $\omega \in \Omega^n(M)$ is compactly supported, then so is $\pi^*\omega$, and

$$\int_{\widetilde{M}} \pi^*\omega = 0.$$

Hint for doing this quickly and elegantly: part (e).

6. Let $\widetilde{M}$ be a manifold and suppose that $F : \widetilde{M} \to \widetilde{M}$ is a smooth involution with no fixed-points. (Thus $F \circ F = \mathrm{id}_M$, and for every $p \in M$, $F(p) \neq p$.) Let $\sim$ be the equivalence relation on $\widetilde{M}$ generated by declaring $p \sim F(p)$. (Thus, the equivalence class of p is the set $\{p, F(p)\}$.) Let $M = \widetilde{M}/\sim$, with the quotient topology.

(a) Show that every smooth involution (whether or not it has any fixed points) is a diffeomorphism.

(b) Show that each $p \in \widetilde{M}$ has an open neighborhood U such that $U \cap F(U) = \emptyset$.

(c) Show that the quotient-construction defining M determines, canonically, a smooth structure on M.

(Idea: Show that $\widetilde{M}$ has an atlas $\tilde{\mathcal{A}}$ such that the domain U of every chart in $\tilde{\mathcal{A}}$ satisfies $U \cap F(U) = \emptyset$. Use such an atlas to construct an atlas $\mathcal{A}$ of M. Show that if we apply this construction to any two atlases of $\widetilde{M}$ [within the given maximal atlas of $\widetilde{M}$] that have the indicated property, that atlases of M we obtain are compatible, and hence determine the same smooth structure on M. The last step is necessary since atlases $\tilde{\mathcal{A}}$ of the type above are not unique.)

For the remainder of this problem, we regard M as a manifold with the above natural smooth structure.

(d) Assume that $\widetilde{M}$ is orientable.

(i) Show that if F is orientation-preserving, then M is orientable.

(ii) Show that if F is orientation-reversing *and $\widetilde{M}$ is connected*, then M is not orientable. (Note that to show that M is not orientable, it's not sufficient to produce a non-oriented atlas! Every manifold, whether or not orientable, has non-oriented atlases.)

Hint: Choose any $\tilde{p} \in \widetilde{M}$. If $\widetilde{M}$ is connected, problem 2 assures us that there is a path in $\widetilde{M}$ from $\tilde{p}$ to $F(\tilde{p})$. Consider the image of this curve under the projection $\widetilde{M} \to M$; note that the curve in $\widetilde{M}$ is a lift of the curve in M. Show that assuming M is oriented leads to a contradiction.

7. Let $n \geq 1$. For each $p \in \mathbf{R}^{n+1}$, let ι_p denote the canonical isomorphism $T_p\mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$. Recall that the *standard orientation of* $\mathbf{R}^{n+1}$ *(as a vector space)* is the orientation-class of the standard basis of $\mathbf{R}^{n+1}$. Regarding $\mathbf{R}^{n+1}$ as a manifold with a one-chart atlas $\{(\mathbf{R}^{n+1}, \text{id.})\}$, we obtain the *standard orientation of* $\mathbf{R}^{n+1}$ *(as a manifold).* (Equivalently, the latter orientation is defined at each $p \in \mathbf{R}^{n+1}$ by using the isomorphism ι_p to pulls back the standard vector-space orientation of $\mathbf{R}^{n+1}$ to an orientation of the vector space $T_p\mathbf{R}^{n+1}$.) These orientations define what we will mean by "positively-oriented" and "negatively oriented" mean for bases of $\mathbf{R}^{n+1}$ and $T_p\mathbf{R}^{n+1}$.

The *standard inner product* on $\mathbf{R}^n$ is the dot-product. At each $p \in \mathbf{R}^{n+1}$, the isomorphism ι_p pulls this inner product back to an inner product on $T_p\mathbf{R}^{n+1}$. Below, these inner products on $\mathbf{R}^{n+1}$ and $T_p\mathbf{R}^{n+1}$ are intended in any reference to orthogonality between vectors and/or subspaces, or to norms.

The *standard unit* $(n+1)$*-disk* is the set $D^{n+1} := \{v \in \mathbf{R}^{n+1} : \|v\| \leq 1\} \subset \mathbf{R}^{n+1}$. As a subset of the manifold $\mathbf{R}^{n+1}$, this disk is a domain with regular boundary. The boundary, of course, is the standard unit sphere S^n. (You should be able to prove these facts easily, but I'm not asking you to do that in this assignment.) Thus, the standard orientation on the manifold $\mathbf{R}^{n+1}$ induces an orientation on $\partial D^{n+1} = S^n$, This defines the *standard orientation of* S^n*.*

Below, for any $p \in S^n$, we regard T_pS^n as a subspace of $T_p\mathbf{R}^{n+1}$.

(a) For each $p \in S^n$, define $N_p \in T_p\mathbf{R}^{n+1}$ by $N_p = i_p^{-1}(p)$. Check that the map $p \mapsto N_p$ is smooth, and hence defines a vector field N along $S^n \subset \mathbf{R}^{n+1}$ (terminology as in problem 1(b)). Check also that $\|N_p\| = 1$.

(b) For each $p \in S^n$, show that T_pS^n is the orthogonal complement of $\text{span}(N_p)$. (Thus N is the *outward-pointing unit normal vector field along* S^n. The outward-pointing property of N is another fact that should be able to prove these facts easily, but that I'm not asking you to prove in this assignment.)

For the remainder of this problem, let $F : S^n \to S^n$ denote the *antipodal map*, i.e the map $p \mapsto -p$. (Note that, for a point p in a general manifold, there is no such thing as "$-p$"; in defining this notation for $p \in S^n$, we are relying on the fact that S^n is a subset of a vector space.)

(c) Check that F is a smooth involution (hence a diffeomorphism) with no fixed-points.

(d) Show that F preserves orientation if n is odd, and reverses orientation if n is even.

For the remainder of this problem, let $M = S^n/\sim$, where the equivalence relation $\sim$ is the one generated by "$p \sim F(p)$", and where M is given the induced smooth structure (see problem 6(c)).

(e) Show that M is diffeomorphic to the projective space $\mathbf{R}P^n = P(\mathbf{R}^{n+1})$, *as defined in last semester's first homework assignment.*[3]

(f) Show that M (and therefore $\mathbf{R}P^n$) is orientable if and only if n is odd.

(g) Show that S^n "is" (more precisely, is diffeomorphic to) the orientation double-cover of $\mathbf{R}P^n$ if and only if n is even. (Part (e) shows that S^n is always *some* double-cover of $\mathbf{R}P^n$, but a general double-cover of a manifold need not be the *orientation* double-cover.)

[3]Another common definition of $\mathbf{R}P^n$ is $S^n/\sim$, but that's not the definition we used. You're showing here that the two definitions yield the same manifold, up to diffeomorphism.