Differential Geometry—MTG 6257—Spring 2022
Problem Set 2
Due-date: Monday, Feb. 28

Required reading: problems 1–8.

Required problems (to be handed in): 2cd, 2e(i), 2e(ii), 2g(ii), 4, 5, 6. In doing any of these problems, you may assume the results of all earlier problems (optional or required). Problems 1–5 were designed to be done *in the given order, without skipping anything*, although problem 3 and problem-parts 1bd don't help with later problems. You should not even *start* on any hand-in part of problem 2 without first reading-through problem1 (at least parts a,c,e) and all prior parts of problem 2.

Optional problems: All the ones that are not required.

1. Let $n \geq 1$, let M be an n-dimensional oriented manifold and let $D \subset M$ be a domain with regular boundary.

(a) Show that $M \setminus D$ is also a domain with regular boundary. (As a set, the boundary coincides with ∂D, of course.)

(b) The submanifold ∂D inherits an orientation from being the boundary of the domain-with-regular-boundary D, and inherits an orientation from being the boundary of the domain-with-regular-boundary $M \setminus D$. How do these orientations compare?

For the remainder of this problem, fix $\omega \in \Omega_c^n(M)$.

(c) Since both D and $M \setminus D$ are domains with regular boundary, both $\int_D \omega$ and $\int_{M\setminus D} \omega$ are defined. Show that

$$\int_M \omega = \int_D \omega + \int_{M\setminus D} \omega. \tag{1.1}$$

(d) Suppose $\omega = d\eta$ for some $\eta \in \Omega^{n-1}(M)$ for which $\text{supp}(\eta) \cap \partial D$ is compact. Observe that Stokes's Theorem can be applied to each of the three integrals in (1.1). Check that your answer to part (c) is consistent with equation (1.1).

(e) Let $p \in M$. For $\epsilon > 0$, let $B_\epsilon(\vec{0})$ and $\bar{B}_\epsilon(\vec{0})$ denote, respectively, the open and closed Euclidean ball of radius ϵ centered at $\vec{0} \in \mathbf{R}^n$. Let (U, ϕ) be a chart of M "centered" at p—i.e. a chart with $p \in U$ and for which $\phi(p) = \vec{0}$. Then $\bar{B}_\epsilon(\vec{0}) \subset U$ for all sufficiently small $\epsilon > 0$. Restricting attention to such $\epsilon > 0$ henceforth, let $D_\epsilon = \phi^{-1}(B_\epsilon(\vec{0}))$. It is easily seen that D_ϵ is a domain with regular boundary (check this, but do not hand it in). Show that $\lim_{\epsilon\to 0} \int_{D_\epsilon} \omega = 0$, and deduce that

$$\int_M \omega = \lim_{\epsilon\to 0} \int_{M\setminus D_\epsilon} \omega. \tag{1.2}$$

2. **(Measure-zero sets in $\mathbf{R}^n$ and manifolds).**

For $\lambda > 0$, let us call any translate of $[0,\lambda]^n = [0,\lambda] \times [0,\lambda] \times \cdots \times [0,\lambda]$ in $\mathbf{R}^n$ (where $n \in \mathbf{N}$ is given) an *n-cube of side λ*. (Equivalently, an n-cube of side λ is a closed ℓ^∞-ball of radius $\lambda/2$ in $\mathbf{R}^n$.) For any n-cube C of side λ, we define the *n-measure* (or *n-volume*) *of C* to be $\mu_n(C) := \lambda^n$. When the intended n is clear from context, we may simply write $\mu(C)$ and call this the *measure* of C.

> **Definition (measure-zero subset of $\mathbf{R}^n$).** A set $Z \subset \mathbf{R}^n$ has *(n-)measure zero* if, for all $\epsilon > 0$, Z can be covered by a countable[1] collection of cubes $\{C_i\}$ for which $\sum_i \mu_n(C_i) < \epsilon$.

(The definition above coincides with the definition of "Lebesgue-measure zero subset of $\mathbf{R}^n$". However, we are *not* doing measure theory here. We are not defining "measurable set", or "measure-(anything other than zero) subset of $\mathbf{R}^n$." The notion of "measure zero" requires no measure theory; it is a much more primitive concept.)

Clearly, if $Z \subset \mathbf{R}^n$ has measure zero, then so does any subset of Z.

(a) Show that if, in the definition of "measure-zero subset of $\mathbf{R}^n$", we use "open n-cubes $(0,\lambda)^n$", exactly the same subsets of $\mathbf{R}^n$ have measure zero. (Hence, at any time later in this problem, if you find it convenient to use covers by open n-cubes instead of closed n-cubes, you may do so.)

(b) Show a *compact* subset $Z \subset \mathbf{R}^n$ has n-measure zero if and only if for all $\epsilon > 0$, Z can be covered by a *finite* collection of cubes $\{C_i\}$ for which $\sum_i \mu_n(C_i) < \epsilon$.

(c) For $k \in \mathbf{N}$ with $k < n$, show that $\mathbf{R}^k \times \{0_{\mathbf{R}^{n-k}}\}$ has n-measure zero.

(d) Show that any countable union of measure-zero subsets of $\mathbf{R}^n$ has measure zero.

Let $U \subset \mathbf{R}^n$ be a nonempty open set and let $F : U \subset \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 map.

(e) Let $K \subset U$ be compact.

(i) Show that there exists a constant $b > 0$ such that if C is a cube of side λ contained in K, then $F(C)$ is contained in a cube of side $b\lambda$. (*Hint*: Lemma 7.2 in the updated "Review of Advanced Calculus" notes that are linked to this semester's class home page.)

(ii) Let $Z \subset U$. Show that if $\mu_n(Z \cap K) = 0$, then $\mu_n(F(Z \cap K)) = 0$.

(iii) Recall that U (or any open subset of $\mathbf{R}^n$) admits an exhaustion by compact subsets: a nested, increasing sequence of compact sets $K_1 \subset K_2 \subset K_3 \subset \dots$ with $\bigcup_{i=1}^\infty K_i = U$. (One such exhaustion can be constructed as follows. Let $\{p_i\}_{i=1}^\infty$

[1] I adhere to the convention countable sets may be finite or countably infinite. However, whether we use this convention or the one in which "countable" means "countably infinite", exactly the same sets subsets of $\mathbf{R}^n$ have measure zero.

be an enumeration of the points in U all of whose coordinates are rational. Choose any norm on $\mathbf{R}^n$, and for each $j \in \mathbf{N}$, let $r_j > 0$ be such that the closed ball $V_j := \bar{B}_{r_i}(p_i)$—as defined by the given norm—lies in U. Then $\bigcup_{j=1}^{\infty} V_j = U$. For each $i \in \mathbf{N}$, let $K_i = \bigcup_{j=1}^{i} V_j$. Then $(K_i)_{i=1}^{\infty}$ is a nested, increasing sequence of compact subsets of U, and $\bigcup_{i=1}^{\infty} K_i = \bigcup_{j=1}^{\infty} V_j = U$.) Use this, and earlier parts of this problem, to show that if $Z \subset U$ and $\mu(Z) = 0$, then $\mu(F(Z)) = 0$.

In other words, a C^1 map carries sets of measure zero to sets of measure zero.

(f) Let $U, V \subset \mathbf{R}^n$ and let $F : U \to V$ be a diffeomorphism. (Recall that our convention is that "diffeomorphism" means "C^∞ diffeomorphism". But for this problem-part, the argument is the same whether we require diffeomorphisms to be C^1, C^∞, or anything in between.) Let $Z \subset U$. Show that $\mu_n(Z) = 0 \iff \mu_n(F(Z)) = 0$.

Definition (measure-zero subset of a manifold). Let M be an n-dimensional manifold with maximal atlas $\mathcal{A}$, and let $Z \subset M$. We say that Z has *measure zero* (or *measure zero in* M), and write $\mu_n(Z) = 0$ (or simply $\mu(Z) = 0$ when no confusion can arise) if for every chart $(U, \phi) \in \mathcal{A}$, $\mu_n(\phi(Z \cap U)) = 0$.

Note that since $\mathbf{R}^n$ is an n-dimensional manifold, we now have two potentially different meanings of "measure zero subset of $\mathbf{R}^n$. Part (g)(i), below, shows that there is no ambiguity.

(g) Let M be an n-dimensional manifold and let $Z \subset M$.

(i) Show that if M has an atlas $\mathcal{A}' = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ (within the implicitly given maximal atlas) such that $\mu_n(\phi_\alpha(Z \cap U_\alpha)) = 0$, then $\mu_n(Z) = 0$.

In other words: if Z can be covered by a collection of chart-domains U_α for which $\phi_\alpha(Z \cap U_\alpha)$ has measure zero, then Z has measure 0. In particular this holds if $M = \mathbf{R}^n$ and $\mathcal{A}'$ is the one-chart atlas $\{(\mathbf{R}^n, \mathrm{id.})\}$.

(ii) Suppose that Z is a submanifold of M with positive codimension. Show that $\mu_n(Z) = 0$.

3. **Topological aspects of measure-zero sets.**

Two facts you may assume to do this problem:

(1) No cube in $\mathbf{R}^n$ has n-measure zero. (2) Every manifold is a *Baire space*, meaning that the countable intersection of open, dense sets is dense.

Some topological terminology to recall: A subset A of a topological space is called *nowhere dense* if its closure $\bar{A}$ has empty interior (equivalently, if $\bar{A}$ contains no nonempty open set).

Clearly if $\bar{A}$ contains no nonempty open set, then neither does A. Hence the complement of a nowhere-dense set is dense.

(a) Show that a *closed* measure-zero subset of a manifold is nowhere dense. (Without "closed", this would be false, as the example of $\mathbf{Q} \subset \mathbf{R}$ shows. Hence the complement of a closed, measure-zero subset of a manifold is open and dense.

(b) Suppose $Z \subset M$ has measure zero and is σ-compact (i.e. the countable union of compact sets). Show that the complement of Z is dense in M.

4. Recall that for a smooth map of manifolds, $F : M \to N$, (i) a critical *point* of F is a point $p \in M$ for which F_{*p} is not surjective; (ii) a critical *value* of F is a point $q \in N$ for which $F^{-1}(\{q\})$ contains a critical point; and (iii) a regular value of F is a point $q \in N$ that is not a critical value. (Note that any point of N that is not in image(F) is automatically a regular value.)

Last semester we proved the *Regular Value Theorem*: If $q \in N$ is a regular value of F, then $F^{-1}(\{q\})$ is a submanifold of M. An important theorem often used in conjunction with the Regular Value Theorem is:

Theorem 1.1 (Sard's Theorem for C^∞ maps[2]) *Let M, N be manifolds and let $F : M \to N$ be smooth. Then the set of critical values of F has measure zero.*

You may assume Theorem 1.1. (For reasons of time, we are not proving it.)

Letting $\mathrm{Crit}(F) \subset M$ denote the set of critical *points* of F, note that Sard's Theorem says *nothing* about $\mathrm{Crit}(F)$ itself; the theorem says only that the *image* $F(\mathrm{Crit}(F))$ has measure zero in N. To illustrate this, consider a constant map $M \to N$, with $\dim(N) > 0$. *Every* point in M is a critical point, but the set of critical *values* is a singleton subset of N, which indeed has measure zero.

Let M be a manifold, $f : M \to \mathbf{R}$ a smooth function. Show that, for almost every $c \in \mathrm{range}(f)$, the set $f^{-1}(c)$ is a codimension-one submanifold of M, and that the *sub-level set* $\{p \in M : f(p) \le c\}$ and *super-level set* $\{p \in M : f(p) \ge c\}$ are domains with regular boundary. Here, "almost every $c \in \mathrm{range}(f)$" means "every $c \in \mathrm{range}(f) \setminus$ (some measure-zero subset of $\mathbf{R}$)."

5. **Measure-zero sets and the computation of integrals on manifolds.**

Suppose M is a compact, oriented manifold of dimension n, and let $\omega \in \Omega^n(M)$. (Compactness has been assumed just to shorten this problem.) Assume we are given explicit formulas for ω—e.g., local-coordinate expressions for ω for each chart in some atlas of M. How do we use this information to *compute* $\int_M \omega$?

As mentioned in class, partitions of unity (POUs) are extremely useful for *defining* integrals n-forms on M, and for proving various theorems. But as a *computational* device, they're completely useless (other than, perhaps, for numerical integration using a computer); their formulas are intractible. Instead, we attempt to choose a collection of *disjoint* charts $\{(U_i, \phi_i)\}$ that cover all but a measure-zero subset of M, and integrate over $\bigcup_i U_i$ without using any POU; in practice we can often find a *single*, convenient chart (U, ϕ) that does the trick. (For example, in a two-chart "stereographic-projection atlas of S^n, each chart-domain covers all but a single point of S^n; in the "standard atlas" of $M = \mathbf{R}P^n$ [respectively, $\mathbf{C}P^n$], each chart-domain covers all of M except for a submanifold diffeomorphic to $\mathbf{R}P^{n-1}$ [respectively, $\mathbf{C}P^{n-1}$].) But in general, $\omega|_{U_i}$ will not have compact support, and the corresponding integral of $(\phi_i^{-1})^*\omega$ over $\mathbf{R}^n$ would be an improper integral. For example, suppose $M = S^n$, and $U = M \setminus \{p := \text{north pole}\}$, and $\phi : U \to \mathbf{R}^n$ is stereographic projection through p, then $\phi(U) = \mathbf{R}^n$. If $\omega|_U = f dx^1 \wedge \cdots \wedge dx^n$ in the corresponding local coordinates, then f will not have compact support unless ω vanishes on an open neighborhood of p. If we use stereographic projection through the *south* pole to define domains D_ϵ "centered" at p as in problem 1(e), then $\phi(M \setminus D_\epsilon) = \bar{B}_{1/\epsilon}(\vec{0})$, the closed ball of radius $1/\epsilon$ centered at the origin. Hence $\int_M \omega = \lim_{\epsilon\to 0} \int_{M\setminus D_\epsilon} \omega = \lim_{R\to\infty} \int_{\bar{B}_R(\vec{0})} (\phi^{-1})^*\omega = \lim_{R\to\infty} \int_{\bar{B}_R(\vec{0})} f$.

The equality "$\int_M \omega = \lim_{R\to\infty} \int_{\bar{B}_R(\vec{0})} f$" in the S^n example is an illustration of the principle that "measure-zero sets don't affect integrals." We have not defined integrals of non-compactly-supported forms (for example, $\omega|_{S^n\setminus\{p\}}$ on the manifold $S^n \setminus \{p\}$), and existence of the single limit $\lim_{R\to\infty} \int_{\bar{B}_R(\vec{0})} f$ is not quite enough to show that $\int_{\mathbf{R}^n} f$ exists (for a general, continuous f)—but if we *did* show that $\int_{\mathbf{R}^n} f$ exists in the given example, then it would be reasonable to make the definition "$\int_{S^n\setminus\{p\}} \omega = \int_{\mathbf{R}^n} f$", and conclude that $\int_{S^n\setminus\{p\}} \omega = \int_M \omega$; i.e. that the value of the integral is unaffected if we delete the measure-zero set $\{p\}$ from M. Below, in a more general setting, you will carry out a version of this argument in way that avoids some technicalities and is sufficient for computation of many integrals. One ingredient of this argument is the following:

Proposition 1.2 ("Smooth Urysohn Lemma") *Let M be a manifold, let $Z \subset M$ be a closed set, and let $U \subset M$ be an open neighborhood of Z. Then there exists a smooth function $\tilde{\chi} = \tilde{\chi}_{Z,U} : M \to [0,1]$, such that $\tilde{\chi}$ is identically 1 on Z and is identically 0 on $M \setminus U$.*

You may assume the "Smooth Urysohn Lemma" below. The proof-strategy is similar to the proof of the "Smooth Tietze Extension Theorem"; I will supply a proof in a solutions handout for the first assignment.

For the remainder of this problem let M be a compact, oriented manifold of dimension n, let $\omega \in \Omega^n(M)$. and let $Z \subset M$ be a closed subset of measure zero.

(a) Show that there exists a sequence $\{K_j \subset M\}_{j=1} \to \infty$, such that (i) $K_1 \supset K_2 \supset K_3 \supset \dots$, (ii) $\bigcap_{j=1}^\infty K_j = Z$; (iii) for each j, the set K_j is a compact domain with regular boundary and (iv) $\lim_{j\to\infty} \int_{K_j} \omega = 0$.

Hint: Apply the "Smooth Urysohn Lemma" to an appropriate, decreasing sequence $(U_j)_{j=1}^\infty$ of open neighborhoods of Z, and use problem 4 for each j.

(b) Show that, for any sequence (K_j) as in (a), $\int_M \omega = \lim_{j\to\infty} \int_{M\setminus K_j} \omega$. Thus, if $M \setminus Z$ is contained in the domain of a chart (U, ϕ), and we define $f : \phi(U) \to \mathbf{R}$ by $(\phi^{-1})^*\omega = f dx_1 \wedge \cdots \wedge dx^n$ in the standard coordinates on $\mathbf{R}^n$, and define $V_j = \phi(U \setminus K_j) = \phi(M \setminus K_j) \subset \mathbf{R}^n$ for each j, then $\int_M \omega = \lim_{j\to\infty} \int_{M\setminus K_j} \omega = \int_{V_j} f$.

Thus we can compute $\int_M \omega$ as the limit of a sequence of integrals $\int_{V_j} f$ on $\mathbf{R}^n$, where $V_1 \subset V_2 \subset V_3 \ldots$. If $\phi(M \setminus Z) = \mathbf{R}^n$ and $\int_{\mathbf{R}^n} f$ exists, then $\int_M \omega = \lim_{j\to\infty} \int_{V_j} f = \int_{\mathbf{R}^n} f$.

6. Let D be a domain with regular boundary in an oriented n-dimensional manifold M, where $n \geq 1$ and let ∂D have the induced orientation. Let $\omega \in \Omega^j(M), \eta \in \Omega^k(M)$, where $j+k = n-1$, and assume that at least one of the sets $\mathrm{supp}(\omega) \bigcap \overline{D}$, $\mathrm{supp}(\eta) \bigcap \overline{D}$, is compact. (Note that the compact-support assumption is superfluous if we assume that M is compact or that $\overline{D}$ is compact.) Prove the "integration-by-parts" formula

$$\int_D d\omega \wedge \eta = \int_{\partial D} \omega \wedge \eta - (-1)^j \int_D \omega \wedge d\eta.$$

Remark. The case $D = M$ (equivalently, $\partial D = \emptyset$) is important all by itself.

7. Let M be a manifold, X a vector field on M, and let $\omega \in \Omega^j(M), \eta \in \Omega^k(M)$. Show that $\iota_X(\omega \wedge \eta) = (\iota_X\omega) \wedge \eta + (-1)^j \omega \wedge \iota_X \eta$.

8. *"Explicit" Poincaré Lemma for star-shaped regions.* The classical Poincaré Lemma asserts that, for all n and all $k > 0$, every closed k-form on $\mathbf{R}^n$ is exact.

Recall that a set U in a vector space is *star-shaped* if there exists $p \in U$ such that for all $q \in U$, the line segment from p to q lies entirely in U. Given such p, we may say that U is "star-shaped with respect to p".[3] In particular, $\mathbf{R}^n$ is star-shaped. In this problem we establish that if U is an open star-shaped subset of $\mathbf{R}^n$, then every closed k-form on U (with $k > 0$) is exact. (Thus the Poincaré Lemma follows as a special case.) There are many ways of showing this; the point of this problem is to give an explicit formula that produces, for each closed form $\omega \in \Omega^k(U)$, a form $\eta \in \Omega^{k-1}(U)$ such that $\omega = d\eta$.

It suffices to produce such a formula under the hypothesis that U is star-shaped with respect to the origin, which we henceforth assume; a more general formula can be obtained from this by applying a translation. The case $n = 0$ is trivial, so we also assume $n > 0$.

Set-up. For $t \in [0, 1]$ define $F_t : \mathbf{R}^n \to \mathbf{R}^n$ by $F_t(x) = tx$. Since U is star-shaped with respect to the origin, $F_t(U) \subset U$. Let V be the vector field $\sum_i x^i \frac{\partial}{\partial x^i}$. For $k > 0$ and $\omega \in \Omega^k(U)$, define $P(\omega) \in \Omega^{k-1}(U)$ by

$$P(\omega) = \int_0^1 t^{-1} F_t^*(\iota_V \omega)\, dt,$$

[3] The set U is *convex* if it is star-shaped with respect to each of its points.

interpreted pointwise:

$$P(\omega)\big|_x = \int_0^1 t^{-1}\left(F_t^*(\iota_V\omega)\right)\big|_x\,dt. \tag{1.3}$$

Despite appearances, this integral is not improper: if we write ω as $\sum_I f_I\,dx^I$, where the sum is over increasing multi-indices of length k, then $(F_t^*(\iota_V\omega))\big|_x = \sum_I t^k f_I(tx)\,\iota_{V_x}dx^I$, so the integrand in (1.3) is $O(t^{k-1})$ as $t \to 0$. **(End of set-up.)**

Your job: Show that if ω is closed, then $\omega = d(P(\omega))$ (and hence that ω is exact).

Remark 1. With $U = \mathbf{R}^3$, we have seen that there is a dictionary translating between "curl of a vector field" (interpreting "vector field" as in Calc 3) and "d of a 1-form", and "between divergence of a vector field" and "d of a 2-form". Given a vector field X such that $\nabla\cdot X = 0$, the map P above (with $k = 2$) provides one way to construct a vector field A such that $X = \nabla\times A$.[4]

Remark 2. As seen in class, for any connected manifold M we have $H^0_{\mathrm{DR}}(M) = \mathbf{R}$. Hence the Poincaré Lemma, generalized to star-shaped regions U as above, can be written as

$$H^k_{\mathrm{DR}}(U) \cong \begin{cases} \mathbf{R} & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases} \tag{1.4}$$

More generally, (1.4) holds under the much weaker assumption that U is *contractible* (you're not allowed to assume this until/unless I have you prove it!), but it is harder to write down an explicit formula analogous to "$P(\omega)$" in that generality.

The next two problems, about de Rham cohomology, are inspired by the presentation in Bott and Tu, *Differential Forms in Algebraic Topology.*

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9. This problem gives a proof that, for any manifold M and any $k \geq 0$, $H^k_{\mathrm{DR}}(M\times\mathbf{R}) \cong H^k_{\mathrm{DR}}(M)$. This fact, plus induction, plus the trivial fact that (1.4) holds for $U = \mathbf{R}^0$, yield another proof of the Poincaré Lemma.

Below, we simply write "H^k" for "H^k_{DR}".

Set-up. Fix a manifold M. For all $(p,t)\in M\times\mathbf{R}$, recall that we can *canonically* identify $T_{(p,t)}(M\times\mathbf{R})$ as $T_pM\oplus T_t\mathbf{R}$. There is a similar canonical identification of cotangent spaces. Hence, letting t denote the standard coordinate on $\mathbf{R}$, there is a well-defined vector field on $M\times\mathbf{R}$ whose value at (p,t_0) is $(0_{T_pM}, \frac{\partial}{\partial t}\big|_{t_0})$, which (with

[4] In case you know the relevant physics: this construction of a vector potential is not terribly useful for E&M, since the regions in which we want to find vector potentials for the magnetic field, e.g. $\mathbf{R}^3$ with a curve removed (for a wire carrying current) are *never* star-shaped.

a slight abuse of notation) we will denote $\frac{\partial}{\partial t}$. Similarly, we have a well-defined 1-form dt on $M \times \mathbf{R}$.

For $k \geq 1$, $\omega \in \Omega^k(M)$, and $(p,t) \in M \times \mathbf{R}$, the value of ω at (p,t) can be written uniquely as $\omega'(p,t) + dt \wedge \omega''(p,t)$, where $\omega'(p,t) \in \bigwedge^k T_p^* M$ and $\omega''(p,t) \in \bigwedge^{k-1} T_p^* M$. (This decomposition may also be characterized by:

$$\omega'(p,t) = s_t^* \left(\iota_{\partial/\partial t}(dt \wedge \omega(p,t))\right), \quad \omega''(p,t) = s_t^* \left(\iota_{\partial/\partial t}\, \omega(p,t)\right), \tag{1.5}$$

where $s_t : M \to M \times \mathbf{R}$ is the map $p \mapsto (p,t)$. You may wish to convince yourself of this by introducing local coordinates $\{x^i\}$ on M. We can then write $\omega(p,t)$ as $\sum_{|I|=k} a_I\, dx^I + \sum_{|J|=k-1} b_J\, dt \wedge dx^J$, where the sums are over increasing multi-indices of the indicated lengths, and where if $k=1$, we interpret the sum over J just as $b\, dt$ for some real number b. Then $\omega'(p,t) = \sum_{|I|=k} a_I\, dx^I$ and $\omega'' = \sum_{|J|=k-1} b_J\, dx^J$, which can be recovered from the coordinate-independent characterization (1.5).)

For each p, the map $t \mapsto \omega''(p,t)$ is a continuous (in fact smooth) function $\mathbf{R} \to \bigwedge^{k-1} T_p^* M$. Hence we can define a linear map $S : \Omega^k(M \times \mathbf{R}) \to \Omega^{k-1}(M \times \mathbf{R})$ by

$$S(\omega)\big|_{(p,t)} = \int_0^t \omega''(p,s)\, ds;$$

for each p the right-hand side is an ordinary Riemann integral of a continuous vector-valued function. For $k=0$, we simply define $S(\omega) = 0$. (**End of set-up**.)

(a) Make sense out of the following formula and show that it is true:

$$d\omega = d_M \omega' + dt \wedge \left(\frac{\partial \omega'}{\partial t} - d_M \omega''\right).$$

(b) Show that for all $\omega \in \Omega^k(M)$,

$$d(S(\omega)) + S(d\omega) = \omega - \pi^* s_0^*\, \omega, \tag{1.6}$$

where $\pi : M \times \mathbf{R} \to M$ is projection onto the first factor, and $s_0 : M \to M \times \mathbf{R}$ is the map $p \mapsto (p,0)$. Consequently, if ω is closed, then $\omega - \pi^* s_0^*\, \omega$ is exact.[5]

(c) Recall that if $F : N_1 \to N_2$ is a map of manifolds, we have $F^*(d\mu) = d(F^*\mu)$ for all differential forms μ on N_2. This implies that, for all $k \geq 0$, the linear map $F^* : \Omega^k(N_1) \to \Omega^k(N_1)$ carries closed forms to closed forms, and exact forms to exact forms, and therefore induces a linear map $H^k(N_2) \to H^k(N_1)$. It is common to denote this map also as F^*, but for clarity in this problem we will denote it as $F^\sharp$.

Show that the "chain rule for pullbacks", $(F \circ G)^* = G^* \circ F^*$, implies that for maps F, G that are composable as indicated, we have $(F \circ G)^\sharp = G^\sharp \circ F^\sharp$. Show also that if $F : N \to N$ is the identity map, then $F^\sharp : H^k(N) \to H^k(N)$ is also the identity (for all k).

[5]Students who've taken algebraic topology will recognize (1.6) as saying that S is a cochain homotopy, between the identity map and the map $\pi^* s_0^*$, on the cochain complex $\Omega^*(M \times \mathbf{R})$.

(d) Letting I denote the identity map $H^k(M \times \mathbf{R}) \to H^k(M \times R)$, use parts (b) and (c) to show that the map $I - \pi^\sharp \circ s_0^\sharp = 0$ (the zero linear map), and hence that $\pi^\sharp \circ s_0^\sharp = I$.

(e) Observing that $\pi \circ s_0$ is the identity map of M, show that $s_0^\sharp \circ \pi^\sharp$ is the identity map $H^k(M) \to H^k(M)$. Combining this with part (d), deduce that *the maps $\pi^\sharp : H^k(M) \to H^k(M \times \mathbf{R})$ and $s_0^\sharp : H^k(M \times \mathbf{R}) \to H^k(M)$ are isomorphisms, and are inverse to each other.*

(f) Show that $H^0(M \times \mathbf{R}) \cong H^0(M)$. Combining this with part (e), we therefore have

$$H^k(M \times \mathbf{R}) \cong H^k(M) \quad \text{for all } k \geq 0. \tag{1.7}$$

10. Let M, N be manifolds, and for $t \in \mathbf{R}$ define $s_t : M \to M \times \mathbf{R}$ by $s_t(p) = (p, t)$. Again let $\pi : M \times \mathbf{R} \to M$ be projection onto the first factor. From problem 8, for each $k \geq 0$ the maps $s_0^\sharp : H^k(M \times \mathbf{R}) \to H^k(M)$ and $\pi^\sharp : H^k(M) \to H^k(M \times \mathbf{R})$ are isomorphisms and are inverse to each other. Similarly, for any $t \in \mathbf{R}$ the map $s_t^\sharp : H^k(M \times \mathbf{R}) \to H^k(M)$ is an isomorphism that inverts $\pi^\sharp$.

Suppose that $F_0, F_1 : M \to N$ are smoothly homotopic maps, i.e. that there exists a smooth map $F : M \times [0, 1] \to N$ such that $F_0 = F \circ s_0$ and $F_1 = F \circ s_1$. ($M \times [0, 1]$ is not a manifold; it is an example of a *manifold-with-boundary*. For the interested student, I have added problem 11 to define "manifold with boundary". But an adequate definition of "smooth map $F : M \times [0, 1] \to N$" that does not require defining manifolds-with-boundary is: For some open neighborhood U of $M \times [0, 1]$ in manifold $M \times \mathbf{R}$, there is an extension of F to a smooth map $U \to N$.) Let $h : \mathbf{R} \to [0, 1]$ be a smooth, monotone function such that $h(t) = 0$ for $t \leq 0$ and $h(t) = 1$ for $t \geq 1$; we saw in the "Bump Function" notes that such functions exist. Define $\tilde{F} : M \times \mathbf{R} \to N$ by $\tilde{F}(p, t) = F(p, h(t))$. Then $\tilde{F}$ is smooth, and its restriction to $M \times [0, 1]$ is simply a reparametrization of the homotopy F. The purpose of introducing $\tilde{F}$ is just to put us in the realm where problem 8 applies directly.

(a) Show that for each $k \geq 0$ we have $F_0^\sharp = F_1^\sharp$ as maps $H^k(N) \to H^k(M)$.

(b) Let $G : M \to N$ be a constant map ($G(M) = \{\text{point}\}$). Show that for $k > 0$, $G^\sharp : H^k(N) \to H^k(M)$ is the zero map, and that for $k = 0$ the map $G^\sharp$ is injective.

(c) M is *smoothly contractible* if the identity map $M \to M$ is smoothly homotopic to a constant map. Show that if M is smoothly contractible, then

$$H^k_{\mathrm{DR}}(M) \cong \begin{cases} \mathbf{R} & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases} \tag{1.8}$$

(This yields yet another proof of the Poincaré Lemma. The map $\mathbf{R}^n \times [0, 1] \to \mathbf{R}^n, (x, t) \mapsto (1 - t)x$, is a homotopy between the identity and a constant map , so $\mathbf{R}^n$ is smoothly contractible.)

Remark: A true fact beyond the scope of this course (because of subject matter, not difficulty) is that if two smooth maps are homotopic, then they are smoothly homotopic. With this fact established, the word "smoothly" can be removed from "smoothly homotopic" and "smoothly contractible" in the above problem.

11. **Manifolds with boundary**. Recall that our original definition of "manifold M" is equivalent to one in which M *starts* as a topological space (rather than inheriting a topology from an atlas), and each chart-map is required to be homeomorphisms from an open set in M to an open subset of $\mathbf{R}^n$ (for some n). For simplicity in this problem, let us assume that we have defined "manifold" this latter way, and that we have required all charts of M to have the same dimension (a condition satisfied automatically if M is connected).

Notation: For $n \geq 1$, let $\mathbf{R}^n_+ = \mathbf{R}^{n-1} \times [0, \infty) \subset \mathbf{R}^n$, which we may call *closed upper half-space* of $\mathbf{R}^n$. We give $\mathbf{R}^n_+$ its induced topology as a subset of $\mathbf{R}^n$.

Note that the boundary of $\mathbf{R}^n_+$, *as a subset of* $\mathbf{R}^n$, is $\mathbf{R}^{n-1} \times \{0\} =: \partial\mathbf{R}^n_+$. We write the interior of $\mathbf{R}^n_+$ (as a subset of $\mathbf{R}^n$) as $\mathrm{Int}(\mathbf{R}^n_+) = \mathbf{R}^{n-1} \times (0, \infty)$.

Below, we always assume $n \geq 1$, and the abbreviation "mwb" stands for "manifold-with-boundary"

A closed domain-with-regular-boundary in a manifold is an example of a *manifold with boundary*, which can be defined by generalizing the definition of *manifold* as follows:

1. Define a *mwb chart* of a topological space M is a pair (U, ϕ), where $U \subset M$ is open, ϕ is a map from $U \to \mathbf{R}^n_+$ for some n, and ϕ is a homeomorphism onto its image $\phi(U)$.

2. If $\hat{U}$ is an open subset of $\mathbf{R}^n$, we call a function $F : \hat{U} \to \mathbf{R}^n$ F *smooth*, or C^∞, if F extends to a smooth map $\tilde{U} \to \mathbf{R}^n$ for some $\mathbf{R}^n$-open neighborhood $\tilde{U}$ of $\hat{U}$. If, in addition, F is a homeomorphism onto its image, and $F^{-1} : F(\hat{U}) \to \hat{U}$ is smooth, we call F a diffeomorphism.

3. An *n-dimensional mwb atlas* on a topological space M is a collection $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ of n-dimensional mwb charts of M with the property that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the "overlap map" $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta a(U_\alpha \cap U_\beta)$ is smooth. (Since this map is automatically bijective and its inverse is a map of the same form with α and β interchanged, any such overlap map is a diffeomorphism.)

   Call a mwb atlas $\mathcal{A}$ on a topological space M *maximal* if $\mathcal{A}$ is not properly contained in another mwb atlas for $\mathcal{A}$. (Equivalently: call two mwb atlases on M *compatible* if their union is a mwb atlases; note that any two such atlases must be of the same dimension. Compatibility is an equivalence relation on the set of mwb atlases of M. A maximal mwb atlas is the union of all mwb atlases compatible with some given mwb atlas.)

4. An n-dimensional *manifold with boundary* is a topological space M together with a maximal mwb atlas on M. Generally we simply call M a manifold-with-boundary, with the understanding that there is a maximal mwb atlas in the background that we don't wish to incorporate into the notation.

Henceforth, for a given manifold-with-boundary, we will simply call a mwb chart a *chart*.

We define "smooth map" from a manifold-with-boundary M to a manifold (or manifold-with-boundary) N just as we did for maps from a *manifold* M to a manifold N: we require the chart-representatives to be smooth maps from open subsets of $\mathbf{R}_+^{\dim(M)}$ to $\mathbf{R}^{\dim(N)}$.

Definition 1.3 Let M be an n-dimensional manifold-with-boundary. We call $p \in M$ a *boundary point* of M if, within the given maximal mwb atlas, there is a chart (U, ϕ) for which $\phi(p) \in \partial\mathbf{R}_+^n = \mathbf{R}^{n-1} \times \{0\}$. The *boundary* of M (in the sense of "manifolds with boundary") is the set of boundary points of M, and is denoted ∂M.

Note that a manifold-with-boundary M is a (true) manifold iff $\partial M = \emptyset$.

Remark 1.4 In the *topological* sense of boundary (a notion defined for subsets of a given topological space), the boundary of every topological space is *empty*. The notation "∂M" for manifolds-with-boundary does *not* represent the topological boundary of M, unless (perhaps) M comes to us as a subset of some larger topological space. An example of the latter type is $\mathbf{R}_+^n$ itself, which has a one-chart mwb atlas. The set $\partial\mathbf{R}_+^n$ that we defined earlier is simultaneously the boundary of the manifold-with-boundary $\mathbf{R}_+^n$, and the boundary of $\mathbf{R}_+^n$ as a subset of $\mathbf{R}^n$.

(a) Suppose $p \in \partial M$, and let (U, ϕ) be a chart of M for which $p \in U$ and $\phi(p) \in \partial\mathbf{R}_+^n$. Let (V, ψ) be another chart of M for which $p \in V$. Show that $\psi(p) \in \partial\mathbf{R}_+^n$ as well.

Thus, for any point p in M, either $\phi(p) \in \partial\mathbf{R}_+^n$ for *every* chart (U, ϕ) with $p \in U$, or for *no* chart (U, ϕ) with $p \in U$.

(There are essentially two ways to do this problem-part. One does not involve differentiability of overlap-maps at all; it simply uses the fact that they are homeomorphisms and applies a fact called *invariance of domain*, a nontrivial result you may have learned in a topology class. The other approach relies on the fact that our overlap-maps are assumed smooth (C^1 would be enough) and applies the Inverse Function Theorem.)

(b) Show that, if $\partial M \neq \emptyset$, then an n-dimensional mwb atlas on M determines, canonically, an $(n-1)$-dimensional *manifold* atlas on M.

(c) Let M be a manifold and let $D \subset M$ be a closed domain with regular boundary. Show that D naturally inherits the structure of a manifold-with-boundary whose boundary ∂D coincides with the topological boundary of D as a subset of M.