Differential Geometry—MTG 6257—Spring 2022
Problem Set 3
Due-date: Monday, Apr. 4

Required reading: problems 1–5 (including any extra reading that a problem's instructions say to do), plus the posted notes "$\mathcal{F}$-linearity, tensoriality, and related notions." Some problems introduce terminology and ideas you'll need in later problems.

Required problems (to be handed in): 3b, 4b, 5b, 8d(i). In doing any of these problems, you may assume the results of all earlier problems (optional or required). You may also use results from notes I've posted for this class (in particular, the notes mentioned above and the notes on tensor products), unless what I'm asking you to prove is one of those results whose proof was left as an exercise.

Optional problems: All the ones that are not required.

1. Read the subsection "Exterior powers $\bigwedge^k(V)$ and their universal property" in the posted notes on tensor products (Section 4.3, as of the time of this writing, but that could change), and do the last exercise in that section (showing, a finite-dimensional vector space V, that $\bigwedge^k(V^*)$ (defined using the notes' definition of $\bigwedge^k$(any vector space)) is canonically isomorphic to $\left(\bigwedge^k(V)\right)^*$ and to the space of alternating functions $\underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbf{R}$. If you need to trace back through the notes for earlier definitions or results, do so.

2. Let V be an n-dimensional vector space, $0 < n < \infty$, let $\{\theta^i\}_{i=1}^n$ be a basis of V^*, and let $k \in \{1, 2, \ldots, n\}$. Show that

$$\left\{\theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_k}\right\}_{1 \le i_1 < i_2 < \cdots < i_k \le n}$$

is a basis of $\bigwedge^k(V^*)$, by (at least) one of the following two methods:

(i) Take the *first* definition of $\bigwedge^k(V^*)$ as the space of multilinear, alternating functions $\underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbf{R}$, and fill in the details of the proof that was sketched in class a long time ago; or

(ii) Proceed directly from the definition of $\bigwedge^k$(any vector space) in the posted notes on tensor product.

3. **Induced connections on dual bundles.** Let ∇ be a connection on a vector bundle E over a manifold M.

(a) In class we sketched a proof of the following result: there is a unique connection ∇' on the dual bundle E^* such that

$$X(\langle \xi, s\rangle) = \langle \nabla'_X \xi, s\rangle + \langle \xi, \nabla_X s\rangle \tag{1.1}$$

for all vector fields X and all sections ξ, s of E, E^* respectively. (In (1.1), the dual-pairings are taken pointwise, of course.) Fill in the details of this sketch.

(b) Let $\{s_\alpha\}_{\alpha=1}^k$ be a local basis of sections of E and let $\{\xi^\alpha\}$ be the local basis of sections of E^* dual to $\{s_\alpha\}$. Let $\Theta, \Theta' \in \Omega^1(U; M_{k\times k}(\mathbf{R}))$ be the connection forms of ∇, ∇' with respect to these local bases. (Here and in later problems, $M_{k\times k}(\mathbf{R})$ denotes the space of $k \times k$ real matrices.) Since we are using upper indices for the basis sections of E^*, we write the first index of Θ' downstairs and the second index upstairs[1]:

$$\nabla' \xi^\beta = \xi^\alpha \otimes {(\Theta')_\alpha}^\beta$$

Show that Θ' is the negative transpose of Θ, in the sense that ${(\Theta')_\alpha}^\beta = -{\Theta^\beta}_\alpha$.

(c) Show that for all $p \in M$ and $X, Y \in T_pM$, the endomorphism $F^{\nabla'}(X,Y) : E_p^* \to E_p^*$ is the negative of the natural adjoint of $F^\nabla(X,Y) : E_p \to E_p$. (For the definition of "natural adjoint", see the Appendix in the posted tensor-product notes.)

4. **Induced connections on direct sums, tensor products, and homomorphism bundles.** Let $\nabla^{(1)}, \nabla^{(2)}$ be connections on vector bundles E_1, E_2 over a manifold M. Let $U \subset M$ be an open set over which both E_1 and E_2 are trivial, let $\mathbf{s} := \{s_\alpha\}_{\alpha=1}^{k_1}, \mathbf{t} := \{t_\mu\}_{\mu=1}^{k_2}$ be bases of sections of E_1, E_2 (respectively) over U, and let $\Theta^{(1)} \in \Omega^1(U; M_{k_1\times k_1}(\mathbf{R})), \Theta^{(2)} \in \Omega^1(U; M_{k_2\times k_2}(\mathbf{R}))$ be the corresponding connection forms. (Your answer should be expressed in terms of $\Theta^{(1)}$ and $\Theta^{(2)}$.)

(a) The *direct sum connection* ∇ on $E_1 \oplus E_2$ is defined by $\nabla \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \nabla^{(1)} s \\ \nabla^{(2)} t \end{pmatrix}$, i.e. $\nabla_X \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \nabla^{(1)}_X s \\ \nabla^{(2)}_X t \end{pmatrix}$, where $s \in \Gamma(E_1), t \in \Gamma(E_2)$, and $X \in \Gamma(TM)$. (It is easy to check that this *does* define a connection.) Find the connection form of ∇ with respect to the basis $\begin{pmatrix} s_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} s_{k_1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ t_{k_2} \end{pmatrix}$.

(b) (i) Show that there is a unique connection ∇ on $E_1 \otimes E_2$ such that

$$\nabla_X(s \otimes t) = (\nabla^{(1)}_X s) \otimes t + s \otimes \nabla^{(2)}_X t \tag{1.2}$$

for all $s \in \Gamma(E_1), t \in \Gamma(E_2)$, and $X \in \Gamma(TM)$. Note that this cannot be deduced from applying the universal property of tensor products to $\Gamma(E_1) \otimes \Gamma(E_2)$, since "$s \otimes t$"

[1] By default, LaTeX stacks superscripts directly on top of subscripts, as in B^k_j, making it impossible to distinguish which is the first index and which is the second. One way to produce, say, ${B^i}_j$, is \$ { B^i}_ j \$.

denotes the *pointwise* tensor product $p \mapsto s(p) \otimes t(p)$. If we attempt to take (1.2) as a *definition* of ∇, it is not obvious without a little computation that $\nabla_X(s \otimes t)$ is well-defined, not just because elements of the form $s_p \otimes t_p$ don't form a basis of the vector space $E_{1,p} \otimes E_{2,p}$, but because $s \otimes t = fs \otimes (1/f)t$ for any nonvanishing $f \in C^\infty(M)$.

We call ∇ the *tensor product connection* determined by $\nabla^{(1)}$ and $\nabla^{(2)}$.

(ii) Find the connection form of ∇ with respect to the local basis of sections $\{s_\alpha \otimes t_\mu\}$ of $E_1 \otimes E_2$, expressed in terms of $\Theta^{(1)}$ and $\Theta^{(2)}$. To write this down efficiently, it's helpful to have an object you can call the "tensor product of two matrices". This object is defined in the handout "Some notes on tensor products"; see the Remark in these notes entitled " 'tensor product' of two matrices". (As of this writing, this Remark resides in Section 2.5, "The finite-dimensional case"', as Remark 2.29. Numbering in these notes is subject to change, since the notes are still a work in progress.) The precise definition is really not important for this exercise; all that really matters is that there *is* a definition that's consistent with standard conventions of linear algebra.

(iii) Show that the curvature F^∇ satisfies

$$F^\nabla(X,Y)(s \otimes t) = (F^{\nabla^{(1)}}(X,Y)s) \otimes t + s \otimes (F^{\nabla^{(2)}}(X,Y)t) \tag{1.3}$$

for all $s \in \Gamma(E_1), t \in \Gamma(E_2)$, and $X, Y \in \Gamma(TM)$. We may write (1.3) symbolically as

$$F^\nabla = F^{\nabla^{(1)}} \otimes \mathrm{id}_{E_2} + \mathrm{id}_{\mathrm{E}_1} \otimes F^{\nabla^{(2)}}.$$

(c) Combining problems 4b and 3, there is an induced connection ∇ on $E_2 \otimes E_1^* \overset{\cong}{\underset{\text{can.}}{}}$ $\mathrm{Hom}(E_1, E_2)$. Show that this connection satisfies

$$(\nabla_X A)(s) = \nabla_X^{(2)}(A(s)) - A(\nabla_X^{(1)}(s))$$

for all $A \in \Gamma(\mathrm{Hom}(E_1, E_2))$, $s \in \Gamma(E_1)$, and $X \in \Gamma(TM)$. Note that this can be written as the Leibnizian-looking formula

$$\nabla_X^{(2)}(A(s)) = (\nabla_X A)(s) + A(\nabla_X^{(1)}(s)).$$

(d) Show that the induced connection ∇ on $\mathrm{End}(E_1) \overset{\cong}{\underset{\text{can.}}{}} E_1 \otimes E_1^*$ satisfies

$$F^\nabla(X,Y)A = [F^{\nabla^{(1)}}(X,Y), A] := F^{\nabla^{(1)}}(X,Y) \circ A - A \circ F^{\nabla^{(1)}}(X,Y)$$

for all $A \in \Gamma(\mathrm{End}(E_1))$ and $X, Y \in \Gamma(TM)$.

(e) Let $\mathbf{R}_M$ denote the product bundle $M \times \mathbf{R} \to M$, and let $\nabla^{(0)}$ denote the canonical connection on $\mathbf{R}_M$. (Recall from class that $\nabla^{(0)} f = df$.) The bundles $\mathbf{R}_M \otimes E_1$ and $E_1 \otimes$ are canonically isomorphic to E_1. Show that these canonical isomorphisms carry the tensor-product connections on $\mathbf{R}_M \otimes E_1$ and $E_1 \otimes \mathbf{R}_M$, induced by $\nabla^{(0)}$ and $\nabla^{(1)}$, to the connection $\nabla^{(1)}$.

Remark 1.1 The construction of direct-sum connections and tensor-product connections extends in an obvious way to direct sums and tensor products of more than two vector bundles. In particular, a connection on a vector bundle E induces a connection on any bundle of the form $E_1 \otimes \ldots \otimes E_k$ ($k \geq 1$) where for each i, the bundle E_i is either E or E^*. It is too cumbersome to have distinct notation for each of these induced connections (as we did in problems 1 and 2). Hence, if ∇ is a connection on E, we generally use the same notation ∇ for the induced connection on any of these bundles. In any term in a formula or equation, context—the type of section being differentiated—makes clear which connection is being used. ■

Remark 1.2 (A convention used below) A *tensor bundle* over a manifold M is any bundle of the form $E_1 \otimes \ldots \otimes E_k$ ($k \geq 1$), where for each i, the bundle E_i is either TM or T^*M; if $k = 1$ we also allow the trivial product bundle $\mathbf{R}_M = M \times \mathbf{R} \to M$ [which was denoted $E^{(0)}$ in class]. (Because of the canonical isomorphisms mentioned in problem 4(e), we gain no new bundles by allowing $E_i = \mathbf{R}_M$ if $k > 1$, but there is no harm in allowing it.) A connection ∇ on TM then induces a connection (also denoted ∇) on every tensor bundle over M, provided we define which connection to use on the trivial bundle $\mathbf{R}_M$. In view of problem 2(e), in the context of induced connections on tensor bundles, we define the "induced" connection ∇ on $\mathbf{R}_M$ to be the canonical connection on this product bundle (no matter what connection is used on TM).

With this convention, given a connection ∇ on TM, the collection of induced connections on tensor bundles is "Leibnizian with respect to contractions" in the sense that (1.1) holds with a single symbol "∇", and "Leibnizian with respect to tensor products" in the sense that (1.2) holds with a single symbol "∇". ■

Remark 1.3 We also sometimes refer to sub-bundles and direct sums of tensor bundles as tensor bundles, but do not make that generalization in this homework assignment. ■

5. Let ∇ be a connection on a vector bundle E over a manifold M. By problems 3 and 4, ∇ induces a connection on $E^* \otimes E^*$.

(a) Show that the induced connection ∇ on $E^* \otimes E^*$ preserves the sub-bundles $\mathrm{Sym}^2(E^*)$ and $\bigwedge^2(E^*)$ in the following sense: if s is a section of either of these sub-bundles, and X is a vector field on M, then $\nabla_X s$ is a section of the same sub-bundle. Thus, the restriction of ∇ to sections of either of these sub-bundles is a connection on that sub-bundle.

(b) Let g be a Riemannian metric (in the vector-bundle sense) on E, and let $\mathsf{g} \in \Gamma(\mathrm{Hom}(E, E^*))$ be the section whose value at $p \in M$ is the isomorphism $\mathsf{g}_p : E_p \to E_p^*$

determined by the metric g. Show that if g is covariantly constant, then so is g:

$$\nabla_X(\mathsf{g}(s)) = \mathsf{g}(\nabla_X s) \quad \text{for all } s \in \Gamma(E), X \in \Gamma(TM).$$

6. **The covariant Hessian.** Let ∇^E be a connection on a vector bundle E over M, and let ∇^M be a connection on TM. For all vector fields X, Y and all $s \in \Gamma(E)$, let

$$(\widetilde{H}s)(X,Y) = \nabla^E_X \nabla^E_Y s - \nabla^E_{\nabla^M_X Y} s.$$

(In case the last term of the formula is hard to read: in that term, "$\nabla^M_X Y$" is a subscript to ∇^E; at each $p \in M$ we are differentiating s in the direction $\nabla^M_X Y\big|_p$, using the connection ∇^E.

(a) Show that $(\widetilde{H}s)(X,Y)$ is $\mathcal{F}$-bilinear in (X,Y). Hence, for each s, the map $(X,Y) \mapsto (\widetilde{H}s)(X,Y)$ is tensorial, and therefore defines a section Hs of $E \otimes T^*M \otimes T^*M$.

The section Hs is called the *covariant Hessian of* s with respect to the connections ∇^E and ∇^M. If (M,g) is Riemannian, and you see the term "covariant Hessian" used without the connections ∇^E and ∇^M having both been specified explicitly, the writer is probably using the following conventions:

- ∇^M is the Levi-Civita connection on (M,g).
- If E is a tensor bundle over M, then ∇^E is the one induced by the Levi-Civita connection. Note that for the product bundle $\mathbf{R}_M$, this means that the canonical connection ($\nabla f = df$) is used, so the covariant Hessian of $f \in C^\infty(M)$ is given by $Hf(X,Y) = X(Y(f)) - (\nabla_X Y)(f)$, where ∇ is the Levi-Civita connection.

(b) Show that if ∇^M is torsion-free, then for all sections s and vector fields X, Y,

$$Hs(X,Y) - Hs(Y,X) = F^{\nabla^E}(X,Y)s.$$

Thus, the left-hand side is tensorial in s, even though neither term individually is tensorial in s.

(c) Following the conventions mentioned above, show that on a Riemannian manifold, the covariant Hessian of any function $f \in C^\infty(M)$ is a *symmetric* tensor field. (All that is needed for this symmetry is that we use a torsion-free connection on TM; the metric does not enter the argument.)

7. **Higher-order covariant derivatives.** Let ∇ be a connection on TM. Using the convention in Remark 1.2, we have an induced connection denoted ∇ on every

tensor bundle over M. Since a connection on any vector bundle E maps $\Gamma(E)$ to $\Gamma(E \otimes T^*M)$, we therefore have an infinite sequence of maps

$$C^\infty(M) = \Gamma(\mathbf{R}_M) \xrightarrow{\nabla = d} \Gamma(T^*M) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M \otimes T^*M) \xrightarrow{\nabla} \dots . \tag{1.4}$$

(For any tensor bundle E we have a similar sequence, with $\Gamma(\mathbf{R}_M)$ replaced by $\Gamma(E)$, and with $(T^*M)^{\otimes k}$ replaced by $E \otimes (T^*M)^{\otimes k}$.) In particular, for $f \in C^\infty(M)$, a connection on TM allows us to define $\nabla\nabla f, \nabla\nabla\nabla f$, etc.

(a) Show that if the connection ∇ is torsion-free, then $\nabla\nabla f$ is the covariant Hessian defined in problem 3.

If you do part (a) carefully, you will likely find that you are actually doing part (b), and then deducing part (a) using problem 5b. I've stated part (a) separately since it's more forgiving of an easily-made mistake.

(b) More generally, show that if the connection ∇ is arbitrary, then $\nabla\nabla f$ is still the covariant Hessian up to a "transpose":

$$(\nabla\nabla f)(X, Y) = Hf(Y, X). \tag{1.5}$$

(This is true with f replaced by a section of any tensor bundle; I'm just giving you the simplest case for homework. An analog is also true for sections of an *arbitrary* vector bundle E, except that we need to specify two initial connections, ∇^E and ∇^M, to define what "∇" is going to mean beyond the first map in the sequence analogous to (1.4).)

8. **Covariant exterior derivative.** Let E be a vector bundle over a manifold M. As in class, we will use the abbreviated notation "$\Omega^j(E)$" for $\Omega^j(M; E) = \Gamma(E \otimes \bigwedge^j T^*M)$.

(a) Let $j, l \geq 0$.

(i) Show that there is a unique bilinear map $\wedge : \Omega^j(E) \times \Omega^l(M) \to \Omega^{j+l}(E)$, $(\alpha, \omega) \mapsto \alpha \wedge \omega$, satisfying

$$(s \otimes \eta)_p \wedge \omega_p = s_p \otimes (\eta \wedge \omega)_p \ \text{ for all } p \in M. \tag{1.6}$$

(ii) Show that there is a unique bilinear map $\Omega^j(\mathrm{End}(E)) \times \Omega^l(E) \to \Omega^{j+l}(E)$ satisfying

$$((A \otimes \eta)_p, (s \otimes \omega)_p) \mapsto A_p(s_p) \otimes (\eta_p \wedge \omega_p) \ \text{ for all } p \in M. \tag{1.7}$$

(In this equation, the endomorphism A_p is applied to the vector $s_p \in E_p$, while the $\bigwedge^* T_p^*M$-factors are wedged together.) Henceforth we omit the subscript p equations like (1.6) and (1.7), understanding that an equation like "$(s \otimes \eta) \wedge \omega = s \otimes (\eta \wedge \omega)$" is to be interpreted as a pointwise statement.

For $F \in \Omega^j(\mathrm{End}(E))$ and $\xi \in \Omega^l(E)$, we will write $F(\xi)$ for the image of (F, ξ) under the map defined pointwise by (1.7). Regrettably, the notation is not self-explanatory, but I know of no wonderful notation for this combined endomorphism-evaluation/wedge-product operation.

(b) Let ∇ be a connection on E. Show that there is a unique linear map $d_\nabla : \Omega^*(E) \to \Omega^*(E)$ that satisfies

$$d_\nabla(s \otimes \omega) = (\nabla s) \wedge \omega + s \otimes d\omega \tag{1.8}$$

for all $s \in \Gamma(E)$, $\omega \in \Omega^j(M)$, $j \geq 0$. We call d_∇ the *covariant exterior derivative* operator determined by ∇.

(c) Notation as in (b). Show that, for $j \geq 0$, the operator $d_\nabla : \Omega^j(E) \to \Omega^{j+1}(E)$ is not $\mathcal{F}$-linear, but that $d_\nabla \circ d_\nabla : \Omega^j(E) \to \Omega^{j+2}(E)$ *is* $\mathcal{F}$-linear.

(d) Let $F^\nabla \in \Omega^2(\mathrm{End}(E))$ be the curvature 2-form of ∇.

(i) Show that for every $s \in \Gamma(E)$, $d_\nabla d_\nabla s = F^\nabla(s)$, where the notation is as in (a)(ii) above (with $j = 0$).

(ii) Show, more generally, that for any $j \geq 0$ and $\xi \in \Omega^j(E)$, $d_\nabla d_\nabla \xi = F^\nabla(\xi)$.

Remark 1.4 Hence for a *flat* connection, the pair $(\Omega^*(E), d_\nabla)$ is a cochain complex, and cohomology is defined. Remember, however, that not every vector bundle admits a flat connection. For those that do, the cohomology groups (in a given degree) defined by different flat connections may not be isomorphic. ■

9. **Torsion and the covariant exterior derivative.** Let M be a manifold. The identity map $I : TM \to TM$ may be viewed as a TM-valued 1-form on M. (Note that for a general vector bundle, there is no analog of this special 1-form.)

Let ∇ be a connection on TM.

(a) Show that

$$d_\nabla I = \tau^\nabla, \tag{1.9}$$

where the torsion tensor-field τ^∇ is viewed as a TM-valued 2-form (just as is $d_\nabla I$). I.e. *the torsion of a connection on TM is the covariant exterior derivative of the "identity 1-form" $I \in \Omega^1(M; TM)$.*

Remark 1.5 Above, we treated I as an element of $\Omega^1(M; TM)$; the object $d_\nabla I$ was then an element of $\Omega^2(M; TM)$. But we may also view I as tensor field on M, a section of the bundle $\mathrm{End}(TM) = \mathrm{End}(TM \otimes T^*M)$. (In terms of bundle-valued differential forms, I is then a element of $\Omega^0(M; \mathrm{End}(TM))$ rather than $\Omega^1(M; TM)$.) From an earlier problem on this assigment, the connection ∇ on TM induces a connection on $\mathrm{End}(TM)$. With this induced connection, treating I as a section of $\mathrm{End}(TM)$, we have $\nabla I = 0 \in \Gamma(\mathrm{End}(TM) \otimes T^*M) = \Omega^1(M; \mathrm{End}(TM))$. ■

(b) Suppose $\tau^\nabla = 0$. Then, viewing I as a TM-valued 1-form, by part (a) we have $d_\nabla d_\nabla I = d_\nabla(0) = 0 \in \Omega^3(M; TM)$. But by problem 8d(ii), we also have $d_\nabla d_\nabla I = F^\nabla(I) \in \Omega^3(M; TM)$. Hence $F^\nabla(I) = 0$.

In particular "$F^\nabla(I) = 0$" holds if ∇ is the Levi-Civita connection for a Riemannian metric g. Use the definition of $F^\nabla(I)$ (plus the equation $F^\nabla(I) = 0$) to derive a symmetry of the Riemann tensor that we have derived by other means.