Differential Geometry—MTG 6257—Spring 2022
Problem Set 4
Due-date: Wednesday, 4/27/22

Homework should either be given to me in person, or slipped under my door, or placed in the basket that's on my door. Your homework is not to be submitted electronically unless I have given you individual permission for that. (That permission has been given to only one student, and I do not expect to give it to any others. Please remember that my extension of the hand-in date from Apr. 20, the original date, to Apr. 27, is an accommodation to students. In return, I expect that you not make *extra* work for me, which is what electronic submission would do.)

Required problems (to be handed in): 1bc, 2bce, 3b, 6ace. In doing any of these problem parts, you may assume the results of all earlier problems and problem-parts (optional or required).

Optional problems: All the ones that are not required.

Required reading: All parts of problems 1–3, and the reading mentioned in the directions for problem 6.

Notation for all the problems: unless otherwise specified, M is an n-dimensional manifold. (Some problems may still state this explicitly.)

1. **Levi-Civita connection on a submanifold.** Let M be an n-dimensional manifold, N a submanifold, and $j : N \to M$ the inclusion map. For any subset $W \subset M$ and any $Y \in \Gamma(TM|_W)$ (a vector field *along* W), we call Y *tangent to* N if $Y_p \in T_pN$ for all $p \in N \cap W$. (Here and below, we identify T_pN with its image in T_pM under the inclusion map j_{*p}. Thus a vector field <u>on</u> N—a section of TN—is identified with a vector field <u>along</u> N that is tangent to N.)

(a) Let $p \in N$, let $\widetilde{U} \subset M$ be an M-open neighborhood of p that is the domain of an N-adapted chart, let $U = N \cap \widetilde{U}$, and let Y be a vector field along U (thus $Y \in \Gamma(TM|_U)$, but Y is not necessarily tangent to N). It is easily seen from the definition of "adapted chart" that the N-open set $N \cap \widetilde{U}$ is closed *as subset of the manifold* $\widetilde{U}$. Hence, problem 1(b) on this semester's Problem Set 1 shows that Y can be extended to a vector field $\tilde{Y}$ on $\widetilde{U}$. We refer to such a vector field $\tilde{Y}$ on the M-open set $\widetilde{U}$ as a *local extension* of Y to M.

Given vector fields X, Y on N, their Lie bracket (commutator) is another vector field on N. Show that "local extensions of vector fields on submanifolds behave well with respect to commutator," in the sense that if $\tilde{X}, \tilde{Y}$ are local extensions of X, Y to

M, both defined on an M-open set $\widetilde{U}$, then $[\tilde{X}, \tilde{Y}] \in \Gamma(TM|_{\widetilde{U}})$ is is a local extension of $[X, Y]$. (Hence, in particular, $[\tilde{X}, \tilde{Y}]$ is tangent to N.)

For the remaining parts of this problem, let g be a Riemannian metric on N.

(b) Let $\pi^{\tan} : TM|_N \to TN$ denote orthogonal projection (i.e. at each $p \in N$, $\pi^{\tan}$ is the g_p-orthogonal projection $T_pM \to T_pN$). Let $p \in N$. Let ∇ be any connection on TM, and let X, Y be vector fields on N. For any local extensions $\tilde{X}, \tilde{Y}$ of X, Y to M, both defined on an M-open set $\widetilde{U}$, and any $p \in N \cap \widetilde{U}$, define

$$(\nabla'_{\tilde{X}} \tilde{Y})_p = \pi^{\tan}\left((\nabla_{\tilde{X}} \tilde{Y})_p\right).$$

Prove that $(\nabla'_{\tilde{X}} \tilde{Y})_p$ is independent of the choices of extensions $\tilde{X}, \tilde{Y}$ of X, Y.

(c) In view of part (b), given ∇ as above we can unambiguously define an operator $\overline{\nabla} : \Gamma(TN) \times \Gamma(TN) \to \Gamma(TN)$ by

$$\overline{\nabla}_X Y = \pi^{\tan}\left(\nabla_{\tilde{X}} \tilde{Y}\right), \tag{1.1}$$

where $\tilde{X}, \tilde{Y}$ are arbitrary (smooth) local extensions of X, Y to N, since at each $p \in N$ the right-hand side of (1.1) is independent of the choices of extensions. Prove that $\overline{\nabla}$ is a connection on TN.

(d) Prove that if the connection ∇ in (d) is the Levi-Civita connection ∇^M of (M, g), then the connection $\overline{\nabla}$ preserves j^*g and is torsion-free, hence is the Levi-Civita ∇^N connection of (N, j^*g). Thus we have the following fact, which simplifies many computations:

$$\nabla^N_X Y = \pi^{\tan}\left(\nabla^M_{\tilde{X}} \tilde{Y}\right), \tag{1.2}$$

where $\tilde{X}, \tilde{Y}$ are arbitrary local extensions of X, Y.

(*Note*: The Levi-Civita connection of a Riemannian manifold, e.g. (M, g) depends on the metric g, not just on the manifold M. In our notation "∇^M" and "∇^N" we are suppressing the metric-dependence just to avoid unwieldy notation.)

2. **Second fundamental form.** Let (M, g) be a Riemannian manifold and N a submanifold. We equip N with the induced metric j^*g, where $j : N \hookrightarrow M$ is the inclusion map. For each $p \in \mathbf{N}$, let $(T_pN)^\perp \subset T_pM$ denote the g_p-orthogonal complement of T_pN. It can be shown that

$$(TN)^\perp := \textstyle\coprod_{p \in N} (T_pN)^\perp \subset TM|_N$$

satisfies the definition of "vector sub-bundle of $TM|_N$" (hence is a vector bundle in its own right). We call $(TN)^{\perp}$ the *geometric normal bundle* of N.

Let ∇^M, ∇^N denote the Levi-Civita connections of (M, g) and (N, j^*g) respectively, and let $\pi^{\rm nor} : TM|_N \to (TN)^{\perp}$ denote the vector-bundle homomorphism that, at each $p \in N$, is the g_p-orthogonal projection $T_pM \to (T_pN)^{\perp}$). (Recall that we previously defined the *algebraic normal bundle* of N to be the vector bundle whose fiber at p is the *quotient* space T_pM/T_pN, rather than any subspace of T_pM. The algebraic normal bundle is defined *without any reliance on a Riemannian metric.* Note that a quotient space is not a subspace; the algebraic normal bundle of N is a *quotient* bundle of $TM|_N$, not a *sub*-bundle of $TM|_N$. However, it is not hard to show that the bundle *homo*morphism $\pi^{\rm nor} : TM|_N \to (TN)^{\perp}$ descends to a bundle *iso*morphism from the algebraic normal bundle to the geometric normal bundle. Hence the geometric normal bundle, whose underlying point-set depends on a metric, is isomorphic to the algebraic normal bundle, whose underlying point-set does not.)

(a) Let $p \in N$, $Y \in T_pN$. Show that there exists an extension of Y to a vector field $\tilde{Y}$ on an M-open neighborhood $\tilde{U}$ of p, with $\tilde{Y}$ tangent to N.

(b) Let $p \in N$, $X, Y \in T_pN$. Let $\tilde{Y}$ be any local extension of Y to M that is tangent to N. (Part (a) says that such an extension exists.) Show that the vector $\pi^{\rm nor}(\nabla^M_X \tilde{Y}) \in (T_pN)^{\perp}$ is independent of the choice of extension $\tilde{Y}$.

(c) In view of part (b), we can unambiguously define a section $\mathbf{h}$ of $\mathrm{Hom}(TN \otimes TN, (TN)^{\perp}) \underset{\text{canon.}}{\cong} T^*N \otimes T^*N \otimes (TN)^{\perp}$ by

$$\mathbf{h}(X, Y) = \pi^{\rm nor}(\nabla^M_X \tilde{Y}), \tag{1.3}$$

where $\tilde{Y}$ is any extension of Y to M that is tangent to N. Show that $\mathbf{h}$ is symmetric: $\mathbf{h}(X, Y) = \mathbf{h}(Y, X)$. (Hence $\mathbf{h}$ is actually a section of the sub-bundle $\mathrm{Sym}^2(T^*N) \otimes (TN)^{\perp}$.)

Remark. Comparing (1.2) and (1.3), we see that for all vector fields $\tilde{X}, \tilde{Y}$ on M that are tangent to N, at each point of N we have

$$\nabla^M_{\tilde{X}} \tilde{Y} = \nabla^N_X Y + \mathbf{h}(X, Y), \tag{1.4}$$

where $X = \tilde{X}|_N$ and $Y = \tilde{Y}|_N$. The first term on the right-hand side is tangent to N and is Leibnizian in Y; the second term is normal to N and is $\mathcal{F}$-linear in Y.

(d) Suppose N has codimension 1. Then, locally, there exist two unit normal vector fields (each the negative of the other). Let ν be one of these unit normal vector fields, say on an open connected set $U \subset N$. Since $(T_pN)^{\perp}$ is 1-dimensional for each $p \in N$, we have

$$\mathbf{h}(X,Y) = g(\mathbf{h}(X,Y),\nu)\nu \tag{1.5}$$

The (scalar-valued) *second fundamental form*[1] *determined by* ν is the tensor field $h \in \Gamma((T^*N \otimes T^*N)|_U)$ given by the coefficient of ν on the right-hand side of (1.5):

$$\begin{aligned} h(X,Y) &= g(\mathbf{h}(X,Y),\nu) \\ &= g(\nabla^M_X \tilde{Y},\nu) \end{aligned}$$

for any extension of Y tangent to N.

Let $\tilde{\nu}$ be any local extension of ν to M. Show that for all $X,Y \in \Gamma(TN|_U)$,

$$h(X,Y) = -g(\nabla^M_X \tilde{\nu}, Y). \tag{1.6}$$

Thus the second fundamental form h measures the *bending of the unit normal.*

Remark. If N is connected and admits a nonvanishing normal vector field, then there are exactly two globally-defined scalar-valued second fundamental forms, each the negative of the other. The tensor field $\mathbf{h} \in \Gamma(\mathrm{Sym}^2(T^*N) \otimes (TN)^\perp)$ can be called the *vector-valued second fundamental form.* Unlike the scalar-valued second fundamental forms, $\mathbf{h}$ is always well-defined globally (and is unique), whether or not N admits a global nonvanishing normal vector field.

In the literature, you will see both h and $\mathbf{h}$ referred to as "the second fundamental form"; you have to tell from context whether the author means the scalar-valued or vector-valued object. (If you are lucky, the author will tell you explicitly.)

(e) Let $M = \mathbf{R}^n$ with the standard metric, and let $N = S^{n-1}$ with the induced metric. Let ν be the outward-pointing normal vector field. Show that the second fundamental form h determined by ν is given by

$$h(X,Y) = -g(X,Y) \quad \text{for all } p \in S^{n-1} \text{ and } X,Y \in T_pS^{n-1}\ .$$

(f) Let $M = \mathbf{R}^3$ with coordinates (x,y,z) and the usual metric, let $U \subset \mathbf{R}^2$ be an open neighborhood of $(0,0)$, let $f : U \to \mathbf{R}$, and let $N \subset \mathbf{R}^3$ be the graph of f. Assume that $f(0,0) = 0$ and that $df|_{(0,0)} = 0$, so that the plane in $\mathbf{R}^3$ tangent to N at $(0,0,0)$ (the "embedded tangent space" at the origin) is the xy plane. Let ν be the upward-pointing normal vector field (i.e. the one whose z-component is positive). Let h be the second fundamental form determined by ν.

[1] Historically, the *first fundamental form* was the induced metric j^*g. Nowadays, the terminology "first fundamental form" has largely been supplanted by the more self-descriptive "induced metric", but the terminology "second fundamental form" has survived unscathed.

The Taylor expansion of f near $(0,0)$ is of the form $f(x,y) = \frac{1}{2}(ax^2+2bxy+cy^2)+$ (higher-order remainder). Express h at the origin in terms of a, b, and c.

Figure out how this generalizes to graphs of functions $f : (U \subset \mathbf{R}^n) \to \mathbf{R}$.

Remark. For any hypersurface (:= codimension-one submanifold) $N \subset \mathbf{R}^n$, and any $p \in N$, we can rotate and translate the coordinate axes in $\mathbf{R}^n$ to make p the origin and T_pN the hyperplane $H = \{x^n = 0\}$. The Implicit Function Theorem then implies that, near the origin, N is the graph of a function $f : (U \subset H) \to \mathbf{R}$. So, your work above provides a general interpretation of the second fundamental form of a hypersurface in $\mathbf{R}^n$: it describes the second-order deviation of the surface from its tangent plane at any point. (There *is* no first-order deviation; that's what "tangent plane" means.) For these reasons, the second fundamental form of a submanifold of $\mathbf{R}^n$ is often called the "extrinsic curvature" of the submanifold; it's something that an observer in $\mathbf{R}^n$, *external* to the submanifold, might describe as "curvature." The Riemann tensor of a submanifold of $\mathbf{R}^n$ is thought of as "intrinsic curvature": once one has the metric on N, nothing involving the ambient manifold is needed to define or compute the Riemann tensor.

3. **Gauss equations in codimension 1.** (a) Let (M, g_M) be Riemannian manifold and let $N \subset M$ be a codimension-1 submanifold. Give N the induced metric, which we denote g_N. Let $p \in N$, let U be a small neighborhood of p in N, let ν be one of the two unit normal vector fields defined on U, and let h be the scalar-valued second fundamental form determined by ν. Thus if $X, Y \in \Gamma(TN|_U)$ and $\tilde{X}, \tilde{Y}$ are extensions of X, Y to a neighborhood of U in M, then

$$\nabla^N_X Y = \nabla^M_{\tilde{X}} \tilde{Y} - h(X,Y)\nu.$$

Use this to establish the *Gauss equations*, initially just at the point p:

$$\begin{aligned} g_N(Riem^N(X,Y)Z,W) &= g_M(Riem^M(X,Y)Z,W) \\ &\quad +h(X,W)h(Y,Z) - h(X,Z)h(Y,W) \end{aligned} \tag{1.7}$$

for all $X, Y, Z, W \in T_pN$. (Note since $Riem^N(X,Y)Z \in T_pN$ and $W \in T_pN$, the left-hand side of (1.7) would have the same value if we replaced "g_N" by "g_M"; above we are choosing to write the left-hand side purely in terms of objects defined on N.) Observe that replacing ν by $-\nu$ has the effect of turning h into $-h$, and therefore has no effect on the right-hand side of (1.7). Thus (1.7) is true globally, where, at any point p, we allow h to be either of the two locally-defined scalar-valued second fundamental forms.

Remark: If (M, g_M) is $(\mathbf{R}^n, g_{\text{Euc}})$ then $Riem^M \equiv 0$, so in this case equation (1.7) expresses $Riem^N$ purely in terms of the second fundamental form.

Remark: Equation (1.7) can also be written using the (unique) vector-valued second fundamental form $\mathbf{h}$:

$$\begin{aligned} g_N(Riem^N(X,Y)Z,W) &= g_M(Riem^M(X,Y)Z,W) \\ &\quad + g_M(\mathbf{h}(X,W),\mathbf{h}(Y,Z)) - g_M(\mathbf{h}(X,Z),\mathbf{h}(Y,W)). \end{aligned} \tag{1.8}$$

Equation (1.8) remains true if we allow N to have arbitrary codimension, but I am not asking you to prove that. (It's not difficult; I'd just rather you directed your time to other problems.)

(b) Let $N = S^n \subset M = \mathbf{R}^{n+1}$ (for this problem-part, $\dim(M) \neq n$). Write "g" for "g_N", and "$Riem$" for "$Riem^N$". Similarly, "Ric" and "R" below are the Ricci tensor and scalar curvature of (S^n, g). Using this and (1.7), find simple formulas for

- $g(Riem(X,Y)Z,W)$ for any vectors X, Y, Z, W are tangent to S^n at the same point;
- all sectional curvatures of (S^n, g) (I have already told you the answer in class, and given you another way to do the computation; you'll just be verifying that tha answer I gave you was correct).

4. Lemma for use in later problem(s). Let $\{y^i\}$ be standard coordinates on $\mathbf{R}^n$, let $\omega \in \Omega^{n-1}(S^{n-1})$ be the standard volume form, and let $\mathrm{Vol}(S^{n-1}) = \int_{S^{n-1}} \omega$ (the volume of the standard, Euclidean, unit sphere). Show that for all $i, j \in \{1, \ldots, n\}$,

$$\int_{S^{n-1}} y^i y^j \ \omega = \frac{1}{n}\,\delta_{ij}\,\mathrm{Vol}(S^{n-1}).$$

(This can be done without any trigonometric integrals.)

Note: $\mathrm{Vol}(S^{n-1})$ can be computed explicitly. I simply am not asking you to do the computation. (However, it can be reduced to a trigonometric integral of the type we teach "reduction formulas" for in Calc 1.)

5. Ricci tensor and scalar curvature. Let (M, g) be a Riemannian manifold. For each $p \in M$ and $X, Y \in T_pM$, the Riemann tensor defines a linear map $T_pM \to T_pM$ by $Z \mapsto R(X,Z)Y$. Define

$$\mathsf{Ric}(X,Y) = \mathsf{Ric}|_p(X,Y) = \mathrm{tr}(Z \mapsto R(X,Z)Y),$$

where "tr" denotes the trace. Thus, if $\{e_i\}$ is an arbitrary basis of T_pM and $\{\theta^i\}$ is the dual basis of T_p^*M,

$$\mathsf{Ric}(X,Y) = \langle \theta^i, R(X,e_i)Y \rangle.$$

Clearly the map $(X,Y) \mapsto \mathsf{Ric}|_p(X,Y)$ is bilinear, so $\mathsf{Ric}|_p$ is an element of $T_p^*M \otimes T_p^*M$. This bilinear form is called the *Ricci tensor* at p. Letting p vary, it is easily seen that $\mathsf{Ric}|_p$ depends smoothly on p, so Ric becomes a tensor field on M, called the *Ricci tensor* (field) or the *Ricci curvature*.

(a) Show that with $p, \{e_i\}, \{\theta^i\}$ as above, the Ricci tensor at p is given by

$$\begin{aligned} \mathsf{Ric} &= R_{jl}\, \theta^j \otimes \theta^l, \\ &\quad \text{where } R_{jl} = R^i{}_{jil} \end{aligned}$$

and where $\{R^i{}_{jkl}\}$ are the components of the Riemann tensor at p with respect to the given bases.

(b) Show that the Ricci tensor is a *symmetric* tensor field: for all $p \in M$ and all $X, Y \in T_pM$, we have $\mathsf{Ric}(X,Y) = \mathsf{Ric}(Y,X)$.

Suggestion: Compute the trace defining $\mathsf{Ric}(X,Y)$ using an orthonormal basis of T_pM. The dual pairing with θ^i then becomes inner product with e_i.

(c) Below, for any normed vector space V, we write $S(V)$ for the unit sphere centered at the origin.

Assume that $n = \dim(M) \geq 2$. Recall that, at each p, the sectional curvature of M at p is a map $G_2(T_pM) \to \mathbf{R}$, $\mathcal{P} \mapsto \sigma(\mathcal{P})$. For $X \in S(T_pM)$ let $X^\perp = \{Y \in T_pM : Y \perp X\}$. Let $G_2^X(T_pM) \subset G_2(T_pM)$ denote the set of all 2-planes in T_pM that contain X. There is a two-to-one map

$$\begin{aligned} \pi_X : S(X^\perp) &\to G_2^X(T_pM), \\ \pi_X(Y) &= \mathcal{P}(X,Y) := \mathrm{span}\{X,Y\}. \end{aligned}$$

(The "two-to-one" comes from the fact that $\pi_X(-Y) = \pi_X(Y)$.) The vector space $X^\perp$ is a Riemannian manifold with the standard Riemannian metric determined by $g_p|_{X^\perp}$; thus $S(X^\perp)$ inherits a Riemannian metric. Orienting $X^\perp$ arbitrarily, and giving S^{n-1} the induced orientation, we then obtain a volume form form ω_{n-2} on $S(X^\perp)$.

(The subscript here is just a reminder of the dimension of $S(X^\perp)$.) Show that for $X \in S(T_pM)$,

$$\int_{S(X^\perp)} (\sigma \circ \pi_X)\ \omega_{n-2} = \int_{S(X^\perp)} \sigma(\mathcal{P}(X,\cdot))\ \omega_{n-2} = \frac{\mathrm{Vol}(S^{n-2})}{n-1}\mathsf{Ric}(X,X). \tag{1.9}$$

Remark 1.1 Hence

$$\frac{1}{n-1}\,\mathsf{Ric}(X,X) = \frac{1}{\mathrm{Vol}(S(X^\perp))}\int_{S(X^\perp)} (\sigma \circ \pi_X)\ \omega_{n-2}\ . \tag{1.10}$$

Thus, up to the normalization constant $\frac{1}{n-1}$, the quantity $\mathsf{Ric}(X,X)$ represents the *average sectional curvature among all two-planes in T_pM that contain X.*[2] ■

Remark 1.2 Recall that for any finite-dimensional vector space V, any symmetric bilinear form $h : V \times V \to \mathbf{R}$ is determined by its restriction to the diagonal: if we know $h(X,X)$ for *all* $X \in V$, then we know $h(X,Y)$ for all $X, Y \in V$. This follows from the *polarization identity*

$$h(X,Y) = \frac{h(X+Y, X+Y) - h(X-Y, X-Y)}{4}\ .$$

Furthermore, if V is equipped with a norm $\|\ \|$, then for all nonzero $X \in V$ we have $h(X,X) = \|X\|^2 h(\hat{X},\hat{X})$, where $\hat{X} = X/\|X\|$. Thus, in the presence of a norm, a symmetric bilinear form h can be completely recovered from the function f_h (notation just for this problem) that h determines on the unit sphere:

[2] The reason we integrated over $S(X^\perp)$ in (1.9) and (1.10), rather than over $G_2^X(T_pM)$, is that $G_2^X(T_pM)$ is diffeomorphic to the projective space $\mathbf{RP}^{n-2}$, which is not orientable when n is even. However, whether or not a Riemannian manifold (N, g_N) is orientable, the metric g_N induces a well-defined measure "$d\mu_N$" on N; it's simply something that we did not discuss in the non-orientable case (it's not a differential form in that case). Therefore for any finite-dimensional inner-product space W, the projectization $\mathbf{P}(W)$ has a Riemannian metric, hence Riemannian measure $d\mu$, induced the by the natural two-to-one covering map $\pi' : S(W) \to \mathbf{P}(W)$ and the standard Riemannian metric on $S(W)$. (Here we regard W as a Riemannian manifold with the standard Riemannian metric determined by the given inner product on W.) Using these facts it can be shown $\mathrm{Vol}(S(X^\perp)) = 2\mathrm{Vol}(G_2^X(T_pM))$ and that

$$\int_{S(X^\perp)} (\sigma \circ \pi_X)\ \omega = \int_{G_2^X(T_pM)} \sigma\ d\mu.$$

Thus (1.10) indeed represents the average value of the function $\sigma|_{G_2^X(T_pM)}$ with respect to the induced measure on $G_2^X(T_pM)$.

$$\begin{aligned} f_h : S(V) := \{X \in V : \|X\| = 1\} &\rightarrow \mathbf{R}, \\ X &\mapsto f_h(X) := h(X, X). \end{aligned}$$

In particular, for each $p \in M$, the function $f_{\mathsf{Ric}} : S(T_pM) \subset T_pM$ carries all the information of the Ricci tensor at p. ■

(d) Let $\mathsf{g}_p : T_pM \to T_p^*M$ be the isomorphism induced by the inner product g_p. For any tensor $h_p \in T_p^*M \otimes T_p^*M$, we define the *trace of h_p with respect to g_p*, denoted $\mathrm{tr}_{g_p}(h_p)$, to be the image of h_p under the following composition

$$T_p^*M \otimes T_p^*M \xrightarrow{\mathsf{g_p}^{-1}\otimes \mathrm{id}} T_pM \otimes T_p^*M \xrightarrow{\text{canon. iso.}} \mathrm{Hom}(T_pM, T_pM) \xrightarrow{\text{trace}} \mathbf{R}.$$

Applying this pointwise to any $h \in \Gamma(\mathrm{Sym}^2(T^*M))$ gives a real-valued function $\mathrm{tr}_g(h) : M \to \mathbf{R}$.

Show that for h as above, $p \in M$, $\{e_i\}$ any basis of T_pM,

$$\mathrm{tr}_g(h)\big|_p = g^{ij}h_{ij} = h^i{}_i = h_i{}^i \ ,$$

where $h_{ij} = h(e_i, e_j)$, $g_{\bullet\bullet}$ is the matrix of g_p with respect to the basis $\{e_i\}$ (i.e. $g_{ij} = g(e_i, e_j)$), and $g^{\bullet\bullet} = (g_{\bullet\bullet})^{-1}$.

(e) The *scalar curvature* or *Ricci scalar* is the real-valued function $\mathsf{R} = \mathrm{tr}_g(\mathsf{Ric})$ on M. Show that at each $p \in M$,

$$\frac{1}{n}\mathsf{R}(p) = \frac{1}{\mathrm{Vol}(S^{n-1})}\int_{S(T_pM)} f_{\mathsf{Ric}}\ \omega_{n-1} \ ,$$

where f_{Ric} is as in Remark 1.2 and ω_{n-1} is the volume form on the sphere $S(T_pM)$ induced by the metric g_p and an arbitrary choice of orientation of T_pM.

Thus, up to the normalization constant $\frac{1}{n}$, the scalar curvature at p is the average value of the function $S(T_pM) \to \mathbf{R}, X \mapsto \mathsf{Ric}(X, X)$. But for each $X \in S(T_pM)$, the quantity $f_{\mathsf{Ric}}(X)$ is itself an average of sectional curvatures (up to a factor of $\frac{1}{n-1}$), so scalar curvature is sometimes thought of as a "double average" of sectional curvatures. However, the word "double" can be eliminated: it can be shown that, up to a dimensional constant, $\mathsf{R}(p)$ is simply the average value of the sectional-curvature function $\sigma_p : G_2(T_pM) \to \mathbf{R}$.

6. Connections on the pulled-back tangent bundle.

Before doing this problem you should read Section 3 of the notes "Pullbacks of Vector Bundles and Connections". Sections 1 and 2 of those notes were sketched in class, but you may find that you need to read portions of these sections in order to understand Section 3. The notation "$F^\sharp$" (with "F" called "f") is defined in Section 2 of these notes.

Let $F : N \to M$ be a smooth map of manifolds. As discussed last semester, a vector field on N does not, in general, push forward to a vector field on M. However, it *does* push forward to a section of the pulled-back tangent bundle: Given $X \in \Gamma(TN)$, we can define a section $\hat{X} \in \Gamma(F^*TM)$ by

$$\hat{X}_p := F_p^\sharp(F_{*p}X_p). \tag{1.11}$$

(a) Let ∇' be an arbitrary connection on $F^*(TM)$ (not necessarily pulled back from a connection on TM). Consider the bilinear, antisymmetric "pseudo-torsion" map $\tilde{\tau}_\psi = \tilde{\tau}_\psi^{\nabla'} : \Gamma(TN) \times \Gamma(TN) \to \Gamma(F^*(TM))$ defined by

$$\tilde{\tau}_\psi(X,Y) = \nabla'_X \hat{Y} - \nabla'_Y \hat{X} - \widehat{[X,Y]}.$$

(The subscript ψ is for "pseudo"; there is no object "ψ" here.)

Show that $\tilde{\tau}_\psi$ is $\mathcal{F}(N)$-bilinear, hence tensorial, defining a section $\tau_\psi = \tau_\psi^{\nabla'} \in \Omega^2(N; F^*(TM))$.

(b) We may view (1.11) as the definition of a canonical $F^*(TM)$-valued 1-form I_ψ on N,

$$I_\psi(X_p) = \hat{X}_p = F_p^\sharp(F_{*p}X_p).$$

Show that $\tau_\psi = d_{\nabla'} I_\psi$.

(c) Show that the condition $\tau_\psi^{\nabla'} \equiv 0$ is equivalent to the statement that for all local-coordinate systems $\{x^i\}$ on N,

$$\nabla'_{\frac{\partial}{\partial x^i}}\left(\widehat{\frac{\partial}{\partial x^j}}\right) = \nabla'_{\frac{\partial}{\partial x^j}}\left(\widehat{\frac{\partial}{\partial x^i}}\right) \quad \text{for all } i,j. \tag{1.12}$$

(d) Show that if ∇' is the pullback of a connection ∇ on TM whose torsion is $\tau = \tau^\nabla$, then $\tau_\psi^{\nabla'} = F^*\tau$, where we define $F^*\tau$ pointwise by

$$(F_p^*\tau)_p(X_p, Y_p) := F_p^\sharp\left(\tau_{F(p)}(F_{*p}X_p, F_{*p}Y_p)\right), \quad p \in N.$$

Hint: Fix an arbitrary point $p \in N$ and let $\{x^i\}, \{y^i\}$ be local coordinates on a neighborhood of $p, F(p)$ respectively. Compute $\tilde{\tau}_\psi \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)$. The Jacobian $\left(\frac{\partial y^i}{\partial x^j} \right)$ will enter your calculation.

(e) Use earlier parts of this problem to show that if ∇ is any torsion-free connection on TM (e.g. the Levi-Civita connection of a Riemannian metric), then (1.12) holds.